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ANTI-REGULAR SEMIGROUPS

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We recall that a *regular* semigroup is one in which every element has at least one inverse — i. e. if $a \in S$, there is an element b in S such that $aba = a$ and $bab = b$. The elements a and b are called *inverses*. We define *anti-regular* semigroups as follows:

Definition: A semigroup S is called *anti-regular* if for each element a in S there is an element b in S such that $aba = b$ and $bab = a$. The elements a and b are then called *anti-inverses* (see [3]).

e. g.

	a	b	c
a	a	a	c
b	a	b	c
c	c	c	a

$$\left.\begin{array}{l} aaa = a \\ bbb = b \end{array}\right\} a \text{ and } b \text{ are their own anti-inverses.}$$

$$ccc = ac = c\} \ c \text{ is its own anti-inverse.}$$

$$\left.\begin{array}{l} cac = cc = a \\ aca = ac = c \end{array}\right\} a \text{ and } c \text{ are anti-inverses.}$$

Lemma 1: *Let S be an anti-regular semigroup. If $a \in S$, then $a^2 = b^2$, where b is any anti-inverse of a. Also $a^5 = a$ for all a in S.*

Proof: Since a and b are anti-inverses then $aba = b$ and $bab = a$. Thus $aa = baba = bb$, and $a^5 = a^2 aa^2 = b^2 ab^2 = b(bab)b = bab = a$.

Theorem 1: *Every anti-regular semigroup is regular, but the converse is false.*

Proof: Let S be a anti-regular semigroup. If $a \in S$, then $a^5 = a$ by lemma 1. Hence $a = a^5 = a(a^3)a \in aSa$ and S is regular.

To see that the converse is not true, consider the regular semigroup (from ([2]):

	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$$\left.\begin{array}{l} bab = bb = c \ (\neq a) \\ bbb = bc = a \ (\neq b) \\ bcb = ba = b \ (\neq c) \end{array}\right\} b \text{ has no anti-inverse in } S.$$

Lemma 2. *Let S be an anti-regular semigroup, and let a and b be anti-inverses in S. The following are equivalent:*

(i) $a^3 = a$

(ii) a^2 *is an idempotent*

(iii) $ab = ba$

Proof: (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii) Assume $a^2 = a^4$. Then $ab = a(aba) = a^2ba = a^4ba \overset{\text{Lemma 1}}{=} b^4ba \overset{\text{Lemma 1}}{=} ba$.

(iii) $\Rightarrow$ (i) $a = bab = bba \overset{\text{Lemma 1}}{=} aaa$.

Theorem 2: *Let S be semigroup Each element in S has a unique anti-inverse (namely itself) in $S \Leftrightarrow S$ is a band* (see [3]).

Proof: ($\Leftrightarrow$) Assume a εS is its own unique anti-inverse in S. Now $a^2(a)a^2 = a^5 \overset{\text{Lemma 1}}{=} a$, and $a(a^2)a = a^3a = aa = a^2 \Rightarrow a^2$ is also an anti-inverse of a in S. Thus $a^2 = a$ and S is a band.

($\Leftarrow$) Assume S is a band and let a S. Then $a = a^2 = a^3 \Rightarrow a$ is an anti-inverse of itself. If b is also an anti-inverse of a, then $b = b^2 \overset{\text{Lemma 1}}{=} a^2 = a$.

Theorem 3: *Let S be a semigroup. Any two elements of S are anti-inverses $\Leftrightarrow S$ is an abelian group in which each element is its own (group) inverse* (see [3]).

Proof: ($\Rightarrow$) Let $a \in S$. Since a is an anti-inverse of itself, then $a^3 = a$, and by Lemma 2) a^2 is an idempotent. Let $a^2 = e$. Assume b is another idempotent in S. Since b is also an anti-inverse of a, then $b = b^2 \overset{\text{Lemma 1}}{=} a^2 = a$. Since S is regular (Lemma 2) and contains only one idempotent, then S is a group [1]. Since any two elements of S are anti-inverses, the commutativity of S also follows from Lemma 2. Each element x in S is its own (group) inverse since $x^2 = e$, for all x in S.

($\Leftarrow$) Let S be an abelian group in which each element is its own (group) inverse, and let e be the identity element in S. If a, and b are any two elements of S, then $a^2 = b^2 = e$. Thus $aba = aab = eb = b$, and similarly $bab = a$, so that a and b are anti-inverses.

Definition: Define a relation ρ on a semigroup S as follows: $a \rho b \Leftrightarrow$ a and b are anti-inverses in S. Let $S[a] = \{x \in S \mid x \rho a\}$, for all a in S.

Theorem 4: If S is a commutative semigroup, the following are equivalent:

(i) S is anti-regular

(ii) ρ is an equivalence relation (actually a congruence) on S

(iii) $S[a]$ is a subsemigroup of S, for all a in S

(iv) $a^3 = a$, for all a in S.

Proof: (i) $\Rightarrow$ (ii) Assume S is anti-regular and let $a \in S$. If b is anti-inverse of a in S, then $aaa \overset{\text{Lemma 1}}{=} abb = bab = a$ and ρ is reflexive. The symmetry of ρ is obvious. To prove transitivity let, a, b, and c be elements of S such that $a \rho b$, and $b \rho c$. By Lemma 1, $a^2 = b^2 = c^2$. Hence $cac = acc = aaa = a$, and similarly $aca = c$ so that ρ is transitive and thus an equivalence relation on S.

To see that ρ is a congruence on S, let a, b, and c be elements of S such that $a \rho b$. Then $ac(bc)ac = abac^3 = bc^3 = bc$, and $bc(ac)bc = babc^3 = ac^3 = ac$ verifies right compatibility. Left compatibility follows by a similar argument.

(ii) $\Rightarrow$ (iii) Assume ρ is an equivalence relation on S and let $a \in S$. By the reflexive property of ρ, $S[a] \neq \varnothing$. Let x, and $y \in S[a]$. Then $xy(a)xy = xax(yy) = a(yy) = aaa = a$, and $a(xy)a = x(aya) = xy$, so that $xy \in S[a]$ and $S[a]$ is a subsemigroup of S.

(iii) $\Rightarrow$ (iv) Let $a \in S$ and assume $x \in S[a]$. Then $x \rho a \Rightarrow a = xax = xxa \overset{\text{Lemma 1}}{=\!=\!=} aaa$.

(iv) $\Rightarrow$ (i) Trivial.

Corollary: *Let S be a commutative semigroup. S is anti-regular $\Leftrightarrow$* $\Leftrightarrow S = \overset{\text{disjoint}}{\bigcup_{a \in S}} S[a]$, *where $S[a]$ is a subsemigroup of S, for all a in S.*

Lemma 3: *Let S be an anti-regular semigroup and let e be an idempotent in S. Then $[S^e] = \{x \in S \mid x^4 = e\}$ is a subgroup of S.*

Proof: $[S^e]$ is closed, for if x, and y are elements of $[S^e]$ then $x^4 = y^4 = e$ and thus $(xy)^5 \overset{\text{Lemma 1}}{=\!=\!=} xy \Rightarrow (xy)(xy)^4 = xy$

$$\Rightarrow (y^3x^3)(xy)(xy)^4 = (y^3x^3)xy$$
$$\Rightarrow y^3x^4y(xy)^4 = y^3x^4y$$
$$\Rightarrow y^3y^4y(xy)^4 = y^3y^4y$$
$$\Rightarrow y^8(xy)^4 = y^8$$
$$\Rightarrow e(xy)^4 = e$$
$$\Rightarrow e(xy)(xy)^3 = e$$
$$\Rightarrow x^4(xy)(xy)^3 = e$$
$$\Rightarrow x^5y(xy)^3 = e$$
$$\overset{\text{Lemma 1}}{=\!=\!=\!\Rightarrow} xy(xy)^3 = e$$
$$\Rightarrow (xy)^4 = e$$
$$\Rightarrow xy \in [S^e]$$

We now show that $[S^e]$ is regular. Let $x \in [S^e]$. Since S is anti-regular x, has an anti-inverse y in S. By Lemma 1, $x^2 = y^2$. Thus $y^4 = y^2y^2 = x^2x^2 = x^4 = e$, and $y \in [S^e]$. It follows that $[S^e]$ is anti-regular and hence regular by Theorem 1.

Now e is the only idempotent in $[S^e]$, for if f is another, then $f = f^4 = e$. Since $[S^e]$ is regular and contains only one idempotent, then $[S^e]$ is a group [1].

Theorem 5: *Let S be a semigroup and let E be the set of idempotents in S. S is anti-regular* $\Leftrightarrow S=\overset{\text{disjoint}}{\bigcup_{e\in E}}[S^e]$, *where $[S^e]$ is a subgroup of S, such that for each x in $[S^e]$ there is a y in $[S^e]$ such that $xy=y^{-1}x=yx^{-1}$ (where x^{-1} is the group inverse of x).*

Proof: ($\Rightarrow$) Assume S is anti-regular and let $x\in S$. Then $x^4=xx^3 \underset{\text{Lemma 1}}{=} x^5x^3=x^8 \Rightarrow x^4$ is an idempotent. Letting $x^4=e$, we have $x\in[S^e]$. If $x\in[S^e]\cap[S^f]$, then $e=x^4=f$. It follows that $S\subseteq \overset{\text{disjoint}}{\bigcup_{e\in E}}[S^e]$. Since the converse is obvious, equality holds. Each $[S^e]$ is a subgroup of S by Lemma 3.

Let $x\in[S^e]$. Since $[S^e]$ is a group, x has an inverse x^{-1} in $[S^e]$ such that $xx^{-1}=x^{-1}x=e$. But, $x^4=e \Rightarrow xx^3=x^3x=e \Rightarrow x^{-1}=x^3$. Since $[S^e]$ is anti-regular (as shown in the proof of lemma 3) then x has an anti-inverse y in $[S^e]$. As shown earlier $y^{-1}=y^3$. Hence $xy=x(xyx)=(xx)yx=y^3x=y^{-1}x$, and similarly $xy=(yxy)y=yx(yy)=yx(xx)=yx^3=yx^{-1}$. Therefore $xy=y^{-1}x= =yx^{-1}$.

($\Rightarrow$) Assume $x\in \overset{\text{disjoint}}{\bigcup_{e\in E}}[S^e]$, and assume that $x\in[S^e]$. By the hypothesis, there is an element y in $[S^e]$ such that $xy=y^{-1}x=yx^{-1}$.

Hence: $xy=y^{-1}x$ and $xy=yx^{-1}$

$$\Rightarrow y(xy)=y(y^{-1}x) \qquad \Rightarrow (xy)x=(yx^{-1})x$$
$$\Rightarrow yxy=ex \qquad \Rightarrow xyx=ye$$
$$\Rightarrow yxy=x \qquad \Rightarrow xyx=y$$

Thus y is an anti-inverse of x and S is anti-regular.

Corollary: *Let S be an inverse semigroup. S is anti-regular $\Leftrightarrow$ S is a semilattice of disjoint (anti-regular) groups.*

Proof: S an inverse semigroup $\Rightarrow E$ is closed and commutative [1].

Corollary: *A group G is anti-regular $\Leftrightarrow$ for each $x\in G$ there is an element y in G such that $xy=y^{-1}x=yx^{-1}$.*

Corollary: *Let S be an anti-regular semigroup. The $\mathcal{H}$ Classes (also the maximal subgroups of S are simply the groups $[S^e]$ of Theorem 5.*

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