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ON REPRODUCTIVE SOLUTIONS OF ARBITRARY EQUATIONS

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The study of general and reproductive solutions of Boolean equations, initiated by Löwenheim [4], [5], was continued and generalized by several authors (cf. bibliography and the literature quoted in [9]). The first results within a purely set-theoretical framework were obtained by Prešić [6], [8] and Božić [1]. The latter author takes a fixed general (reproductive) solution and shows that a function g is a general (reproductive) solution if and only if it is of the form $g=f\varphi$, where φ is a solution of the functional equation $f=f\varphi\psi f$ ($f=f\varphi f$). In this paper we prove that the general (reproductive) solutions g are characterized by the simpler functional equations $f=gh$ ($f=gf$), which we solve.

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Let the sets S, T fulfil $\varnothing\neq S\subseteq T$; the elements of S are called *solutions*. By a *general solution* is meant a mapping $f:T\to T$ such that $f(T)=S$; if, moreover, $f|S=1_S$ (i. e., if $f(s)=s$ for every $s\in S$), then f is called a *reproductive solution*.

Proposition 1. *Let f be a general solution and $g:T\to T$. Then g is a general solution if and only if $g(T)\subseteq S$ and $f=gh$ for some $h:T\to T$.*

Proof. Assume g is a general solution. Then $g(T)\subseteq S$ is fulfilled as an equality. For every $t\in T$, we have $f(t)\in S=g(T)$; choose $x\in T$ such that $g(x)=f(t)$ and set $h(t)=x$. It follows that $gh(t)=g(x)=f(t)$ for every $t\in T$. Conversely, assume $g(T)\subseteq S$ and $gh=f$. Then for every $s\in S$, taking $t\in T$ such that $s=f(t)$, we get $s=g(h(t))$.

A second proof of Proposition 1 can be obtained via [1].

Proposition 2. *Let f be a general solution and g a reproductive solution. Then $f=gf$.*

Proof. For every $t\in T$, we have $f(t)\in S$, hence $g(f(t))=f(t)$.

P r o p o s i t i o n 3. *Let f be a reproductive solution and assume $g: T \to T$ fulfils $g(T) \subseteq S$ and $f = gf$. Then g is a reproductive solution.*

P r o o f. The function g is a general solution by Proposition 1. Moreover, for every $s \in S$ we have $s = f(s)$, hence $g(s) = g(f(s)) = gf(s) = f(s) = s$.

In view of Propositions 1—3, the determination of all general solutions and of all reproductive solutions is now reduced to the solution of the functional equations $f = gh$ and $f = gf$, respectively. This is done in Propositions 4 and 5 below.

P r o p o s i t i o n 4. *Let $f, h: T \to T$. Then:*

(i) *The equation $f = gh$ is consistent if and only if*

$$(\forall x, x' \in T) \;\; h(x) = h(x') \Rightarrow f(x) = f(x'). \tag{1}$$

(ii) *When condition* (1) *is fulfilled, all the solutions g to $f = gh$ are given by the following formula:*

$$g(t) = \begin{cases} f(x), & \text{if } t = h(x); \\ \text{arbitrary}, & \text{otherwise.} \end{cases} \tag{2}$$

P r o o f. Suppose the equation $f = gh$ has a solution g. Then $h(x) = h(x')$ implies $f(x) = g(h(x)) = g(h(x')) = f(x')$. Conversely, assume (1). Then formula (2) defines unambiguously a mapping g for which $g(h(x)) = f(x)$ $(\forall x \in T)$, i. e., $gh = f$. Conversely, $gh = f$ implies that if $t = h(x)$, then $g(t) = gh(x) = f(x)$, i. e., g fulfils (2).

C o r o l l a r y. *Let $f: T \to T$. Then all the solutions of the equation $f = gf$ are given by the following formula:*

$$g(t) = \begin{cases} t, & \text{if } t \in f(T); \\ \text{arbitrary}, & \text{otherwise.} \end{cases} \tag{3}$$

P r o p o s i t i o n 5. *Let $f: T \to T$. Then there is a bijection between all functions $h: T \to T$ satisfying* (1) *and all couples $(\mathcal{H}, \chi)$, where $\mathcal{H}$ is a partition of T, finer than $\{f^{-1}(y)\}_{y \in f(T)}$, and $\chi: \mathcal{H} \to T$ is an injection.*

Recall that a partition $\mathcal{P}$ is said to be finer than a partition $\mathcal{P}'$ on the same set if every $\mathcal{P}$-coset is included in a $\mathcal{P}'$-coset. Clearly $\mathcal{F} = \{f^{-1}(y)\}_{y \in f(T)}$ is a partition of T.

P r o o f. (i) Let h be a function fulfilling (1). Then obviously $\mathcal{H} = \{h^{-1}(y)\}_{y \in h(T)}$ is a partition finer than $\mathcal{F}$, while $\chi: \mathcal{H} \to T$ defined by $\chi(h^{-1}(y)) = y$ $(\forall y \in h(T))$ is obviously an injection.

(ii) Let $\mathcal{H}$ be a partition finer than $\mathcal{F}$ and $\chi: \mathcal{H} \to T$ an injection. For every $x \in T$, set $h(x) = \chi(H)$, where H is the $\mathcal{H}$-coset that contains x; then $h: T \to T$ fulfils (1), because $h(x) = h(x')$ means $\chi(H) = \chi(H')$, hence $H = H'$ that is x and x' belong to the same $\mathcal{H}$-coset, therefore they belong to the same $\mathcal{F}$-coset, say $f^{-1}(y)$, and this means $f(x) = y = f(x')$.

(iii) Under the hypotheses and notation from (i), the function h' constructed from $(\mathcal{H}, \chi)$ as in (ii), coincides with h. For given $x \in T$, the $\mathcal{H}$-coset containing x is that coset $h^{-1}(y)$ for which $h(x)=y$, therefore $h'(x)= =\chi(h^{-1}(y))=y=h(x)$.

(iv) Under the hypotheses and notation from (ii), we have to prove that the couple $(\mathcal{H}', \chi')$ constructed from h as in (i), coincides with $(\mathcal{H}, \chi)$. So $\mathcal{H}'=\{h^{-1}(y)\}_{y\in h(T)}$ for the function h constructed at (ii), while $\chi'(h^{-1}(y))=y$ $(\forall y \in h(T))$.

Let $H \in \mathcal{H}$ and set $\chi(H)=y$. Then $x \in H$ implies $h(x)=\chi(H)=y\in h(T)$ and conversely, if $x \in h^{-1}(y)$, then the $\mathcal{H}$-coset H' of x fulfils $\chi(H')=h(x)= =y=\chi(H)$, therefore $H'=H$, hence $x \in H$. We have thus proved that $H= =h^{-1}(y) \in \mathcal{H}'$.

Let $h^{-1}(y) \in \mathcal{H}'$. Then $y \in h(T)$; take $x \in h^{-1}(y)$ and let H be the $\mathcal{H}$-coset of x. For every $x' \in H$ we have $h(x')=\chi(H)=h(x)=y$, therefore $x' \in h^{-1}(y)$. Conversely, let $x' \in h^{-1}(y)$ and let H' be the $\mathcal{H}$-coset of x'; then $\chi(H')= =h(x')=y=h(x)=\chi(H)$, therefore $H'=H$, hence $x' \in H$. We have thus proved that $h^{-1}(y)=H \in \mathcal{H}$.

We have thus established that $\mathcal{H}'=\{h^{-1}(y)\}_{y\in h(T)}=\mathcal{H}$. Now let $H= =h^{-1}(y)\in\mathcal{H}$. Then $\chi'(h^{-1})(y))=y$ and on the other hand, taking $x \in H$ it follows that $\chi(H)=h(x)=y$, that is $\chi'(H)=\chi(H)$.

We conclude with the remark that the properties of general and reproductive solutions within the set-theoretical framework do not apply to general and reproductive solutions of Boolean equations because the latter are required to be not merely mappings, but Boolean functions.

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