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GENERALIZED CESÀRO NUMBERS

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Let the real sequence $p=(p_n)$ satisfy

$$p_1=1,\ p_n\geqslant 0\ (n\geqslant 2)^{1)} \tag{1}$$

and put $P_n=\sum_{\nu=1}^{n} p_\nu$.

For every real sequence $s=(s_n)$ consider the linear operator $T_\lambda : s\to\tau(\lambda)$, where $\tau(\lambda)=(\tau_n(\lambda))$ is defined by

$$\tau_n(\lambda)=s_n+\frac{\lambda}{P_n}\sum_{\nu=1}^{n} p_\nu s_\nu \qquad (\lambda \text{ real}). \tag{2}$$

We have shown [1] that for $\lambda\neq -P_k/p_k$ $(k=1, 2, \ldots)$ the inverse operator T_λ^{-1} is defined by[2)]

$$s_n=\tau_n(\lambda)-\frac{\lambda}{P_n a_n(\lambda)}\sum_{\nu=1}^{n} p_\nu a_{\nu-1}(\lambda)\,\tau_\nu(\lambda), \tag{3}$$

where

$$a_0(\lambda)=1,\ a_n(\lambda)=a_n(\lambda;\,p)=\prod_{\nu=1}^{n}\left(1+\lambda\frac{p_\nu}{P_\nu}\right)\ (n\geqslant 1). \tag{4}$$

If $P_n\to\infty\,(n\to\infty)$ and $\lambda>-1$, T_λ^{-1} satisfies Toeplitz's conditions, and so, by the theorem of Toeplitz, one can obtain the mercerian theorem for the weighted means $P_n^{-1}\sum_{\nu=1}^{n} p_\nu s_\nu$ of the sequence s. Moreover, if p satisfies the supplementary condition $p_n/P_n\to 0$ $(n\to\infty)$, the hypothesis $\lambda>-1$ is even necessary. This and related theorems have been proved in [1] and [2].

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1) It is essential that p_1 is positive only.

2) Here, the notation is slightly changed, as compared to that used in [1]: one has to substitute in [1] p_n by p_{n+1} and similarly to do with P_n, s_n and $\tau_n(\lambda)$, but not with $a_n(\lambda)$.

Two particular cases of the sequence $a_n(\lambda;\, p)$ are of special interest: (i) if $p=(1)$, $a_n(\lambda;\, p)$ reduces to the so called Cesàro numbers $\binom{\lambda+n}{n}$, which have many interesting properties (see, for example, A. Zygmund [3], v. I, Ch. III, §1); (ii) if $p_1=1$, $p_n=2^{n-2}\,(n\geqslant 2)$, then $a_n(2\mu-2;\, p)=(2\mu-1)\mu^{n-1}$. So it may be of some interest to examine the properties of generalized Cesàro numbers $a_n(\lambda;\, p)$ for various classes of sequences p.

To avoid trivialities, we shall make two restrictions on p: (i) $p_n>0$ for infinitely many indices, for, otherwise, $a_n(\lambda;\, p)$ takes the same value for all n large enough; (ii) $\lambda\neq -P_k/p_k$ $(k=1,\, 2,\, \ldots)$, for, otherwise, $a_n(\lambda)=0$ for $n>k$. Evidently, $a_n(0)=1$ for each n.

From (4) it is immediate

$$P_n a_n(\lambda)=(1+\lambda)\prod_{\nu=1}^{n-1}\left(1+(1+\lambda)\frac{p_{\nu+1}}{P_\nu}\right)\quad\left(n\geqslant 1;\ \prod_{\nu=1}^{0}=1\right). \tag{5}$$

From (4) and (5), respectively, it follows that

$$a_n(\lambda)-a_{n-1}(\lambda)=\lambda\frac{p_n}{P_n}a_{n-1}(\lambda)\qquad (n\geqslant 1), \tag{6}$$

$$P_n a_n(\lambda)-P_{n-1}a_{n-1}(\lambda)=(1+\lambda)p_n a_{n-1}(\lambda)\ \ (n\geqslant 1;\ P_0=0) \tag{7}$$

and, by summation,

$$a_n(\lambda)=1+\lambda\sum_{\nu=1}^{n}\frac{p_\nu}{P_\nu}a_{\nu-1}(\lambda), \tag{6'}$$

$$(n\geqslant 1)$$

$$P_n a_n(\lambda)=(1+\lambda)\sum_{\nu=1}^{n}p_\nu a_{\nu-1}(\lambda). \tag{7'}$$

Suppose that p satisfies (1) and let $\Lambda=\overline{\lim\limits_{n\to\infty}}\, p_n/P_n$. We remark that $0\leqslant\Lambda\leqslant 1$ and that, if $\lambda<0$, $\underline{\lim\limits_{n\to\infty}}\left(1+\lambda\frac{p_n}{P_n}\right)=1+\lambda\Lambda$. Then:

1° $a_n(\lambda)>0$ for $\lambda>-1$. (Follows from (5)). If $-1/\Lambda<\lambda<-1$, $a_n(\lambda)$ is of constant sign for n large enough and if $\lambda<-1/\Lambda$, $a_n(\lambda)$ alters the sign infinitely many times. (In the product (4), there are then finitely or infinitely many negative factors respectively).

2° $a_n(\lambda)$ increases if $\lambda>0$, decreases if $-1<\lambda<0$ and, for n large enough, increases (decreases) through negative (positive) values if $-1/\Lambda<\lambda<-1$. (This follows from (6) on account of 1°).

3° $P_n a_n(\lambda)$ increases if $\lambda>-1$ and, for n large enough, increases (decreases) through negative (positive) values if $-1/\Lambda<\lambda<-1$. (Follows from (7) on account of 1°).

Suppose now that p satisfies, in addition to (1), the condition

$$P_n\to\infty\quad (n\to\infty). \tag{8}$$

Then:

4° $a_n(\lambda)\to\infty$ $(n\to\infty)$ if $\lambda>0$. (Since by 2°, $a_n(\lambda)\geqslant a_0(\lambda)=1$ $(n\geqslant 1)$, (6') implies

$$a_n(\lambda)\geqslant 1+\lambda\sum_{\nu=1}^{n}\frac{p_\nu}{P_\nu},$$

and one has to apply the theorem of Abel-Dini:

$$\left.\sum_{\nu=1}^{\infty}p_\nu=\infty\Rightarrow\sum_{\nu=1}^{\infty}\frac{p_\nu}{P_\nu}=\infty\right).$$

5° $a_n(\lambda)\to 0$ $(n\to\infty)$ if $-1/\Lambda<\lambda<0$. (On account of 1° we may suppose that $a_n(\lambda)$ is positive and decreasing for $n\geqslant N$.[3] So $a_n(\lambda)$ converges to $a(\lambda)\geqslant 0$. Assume that $a(\lambda)>0$. Since $a_n(\lambda)\geqslant a(\lambda)$ $(n\geqslant N)$ and $\lambda<0$, from (6') it follows that

$$a_n(\lambda)\leqslant 1+\lambda\sum_{\nu=1}^{N-1}\frac{p_\nu}{P_\nu}a_{\nu-1}(\lambda)+\lambda\, a(\lambda)\sum_{\nu=N}^{n}\frac{p_\nu}{P_\nu}\quad(n\geqslant N),$$

and by the Abel-Dini theorem $a_n(\lambda)\to-\infty$ $(n\to\infty)$, what contradicts our assumption. Hence, $a(\lambda)=0$).

6° $P_n a_n(\lambda)\to\infty$ $(n\to\infty)$ if $\lambda>-1$. (Since by 3°, $P_n a_n(\lambda)\geqslant P_1 a_1(\lambda)$ $(n\geqslant 1)$, (7') implies

$$P_n a_n(\lambda)=(1+\lambda)\left(p_1+\sum_{\nu=2}^{n}\frac{p_\nu}{P_{\nu-1}}P_{\nu-1}a_{\nu-1}(\lambda)\right)$$

$$\geqslant(1+\lambda)\left(p_1+P_1 a_1(\lambda)\sum_{\nu=2}^{n}\frac{p_\nu}{P_{\nu-1}}\right)$$

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and one has to apply the theorem of Abel-Dini).

7° $P_n a_n(\lambda)\to 0$ $(n\to\infty)$ if $-1/\Lambda<\lambda-1$. Suppose that $P_n a_n(\lambda)$ is positive and decreases for $n\geqslant N$.[4] So $P_n a_n(\lambda)$ converges to $b(\lambda)\geqslant 0$. Assume that $b(\lambda)>0$. Since $P_n a_n(\lambda)\geqslant b(\lambda)$ $(n\geqslant N)$ and $1+\lambda<0$, from (7') it follows that

$$P_n a_n(\lambda)=(1+\lambda)\sum_{\nu=1}^{N-1}p_\nu a_{\nu-1}(\lambda)+(1+\lambda)\sum_{\nu=N}^{n}\frac{p_\nu}{P_{\nu-1}}P_{\nu-1}a_{\nu-1}(\lambda)$$

$$\leqslant(1+\lambda)\sum_{\nu=1}^{N-1}p_\nu a_{\nu-1}(\lambda)+(1+\lambda)\,b(\lambda)\sum_{\nu=N}^{n}\frac{p_\nu}{P_{\nu-1}}$$

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[3] A symmetric reasoning applies if $a_n(\lambda)$ is negative and increasing for $n\geqslant N$.

[4] The same remark as in 3).

and by the Abel-Dini theorem $P_n a_n(\lambda) \to -\infty$ $(n \to \infty)$, what contradicts our assumption. Hence, $b(\lambda) = 0$).

We remark that one could obtain the properties $4° - 7°$ of the sequence $a_n(\lambda)$ by applying the general results about infinite products of the form $\Pi(1+c_n)$ $(c_n \geqslant 0)$ and $\Pi(1-c_n)$ $(0 \leqslant c_n < 1)$ on (4) and (5) (see e. g. K. Knopp [4], Chapter VII, § 28, Theorem 3 and Theorem 4, combined with Remarks and Examples 2).

To obtain the asymptotic behaviour of $a_n(\lambda)$ $(n \to \infty)$, we need some prelimary results.

THEOREM 1 (Cesàro [5]). *If the real sequence p satisfies* (1), (8) *and*

$$\frac{p_n}{P_n} \to 0 \quad (n \to \infty) \tag{9}$$

then

$$\sum_{\nu=1}^{n} \frac{p_\nu}{P_\nu} \cong \log P_n \quad (n \to \infty).$$

THEOREM 2. *Suppose that p satisfies* (1) *and* (8).

(i) *Then the sequence*

$$x_n \overset{\text{def}}{=} \sum_{\nu=2}^{n} \frac{p_\nu}{P_{\nu-1}} - \log P_n$$

is positive, increasing and converges to a number $C(p)$ if and only if

$$\sum_{\nu=2}^{\infty} \frac{p_\nu^2}{P_{\nu-1} P_\nu} < \infty. \tag{10}$$

For $p=(1)$ and $p=(n)$, $C(p)$ reduces to the Euler constant C and to $2C+\log 2$ respectively.

(ii) *If* (10) *holds, then the sequence*

$$x_n' \overset{\text{def}}{=} \sum_{\nu=2}^{\infty} \frac{p_\nu^2}{P_{\nu-1} P_\nu} + \sum_{\nu=2}^{n} \frac{p_\nu}{P_\nu} - \log P_n$$

is positive, decreasing and converges to $C(p)$.

PROOF. Let

$$y_n \overset{\text{def}}{=} \sum_{\nu=2}^{\infty} \frac{p_\nu^2}{P_{\nu-1} P_\nu} - x_n' = \sum_{\nu=2}^{n} \left(\log \frac{P_\nu}{P_{\nu-1}} - \frac{p_\nu}{P_\nu}\right). \tag{11}$$

Since

$$x_n = \sum_{\nu=2}^{n} \left(\frac{p_\nu}{P_{\nu-1}} - \log \frac{P_\nu}{P_{\nu-1}}\right), \tag{12}$$

one has

$$\sum_{\nu=2}^{n} \frac{p_\nu^2}{P_{\nu-1} P_\nu} = x_n + y_n. \tag{13}$$

If $x = P_n/P_{n-1}$ for each $n \geqslant 2$, the inequality

$$1 - \frac{1}{x} \leqslant \log x \leqslant x - 1 \quad (x > 0) \tag{14}$$

reduces to

$$\frac{p_n}{P_n} \leqslant \log \frac{P_n}{P_{n-1}} \leqslant \frac{p_n}{P_{n-1}}.$$

Hence, on account of (12) and (11_2), x_n and y_n are positive and increasing, and, by (11_1), x_n' is decreasing.

Integrating $1 + 1/t^2 \geqslant 2/t$ from 1 to x, one obtains

$$\log x \leqslant \frac{1}{2}\left(x - \frac{1}{x}\right) \quad (x \geqslant 1). \tag{15}$$

Write (15) in the form

$$\log x - 1 + \frac{1}{x} \leqslant x - 1 - \log x \quad (x \geqslant 1)$$

and put $x = P_n/P_{n-1}$. This gives

$$\log \frac{P_n}{P_{n-1}} - \frac{p_n}{P_n} \leqslant \frac{p_n}{P_{n-1}} - \log \frac{P_n}{P_{n-1}},$$

and, by summation, $y_n \leqslant x_n$.

Since, by $0 \leqslant y_n \leqslant x_n$, (13) implies

$$x_n \leqslant \sum_{\nu=2}^{n} \frac{p_\nu^2}{P_{\nu-1} P_\nu} \leqslant 2\, x_n,$$

part (i) of theorem 2 follows from (1), (10) and the monotony of x_n.

If (10) holds, then by (11_1) and (13),

$$x_n' = x_n + \sum_{\nu=n+1}^{\infty} \frac{p_\nu^2}{P_{\nu-1} P_\nu}.$$

Hence, x_n' is positive, and by part (i), x_n' converges to $C(p)$.

THEOREM 3. *Suppose that the sequence* p *satisfies* (1), (8), *and* (10). *Then, for* $\lambda \neq -P_k/p_k$ $(k = 1, 2, \ldots)$ *there exists* $\Gamma(\lambda + 1;\, p)$ (*independent of* n) *such that*

$$a_n(\lambda) \cong \frac{P_n^\lambda}{\Gamma(\lambda + 1;\, p)} \quad (n \to \infty).$$

If $p = (1)$, $\Gamma(\lambda;\, p)$ reduces, by Euler's definition, to the gamma function $\Gamma(\lambda)$. So, the asymptotic relation of theorem 3 may be interpreted as the definition of a generalized gamma function.

PROOF. Since $\log(1+u)=u+O(u^2)$ for small $|u|$, one has by part (ii) of Theorem 2,

$$\log a_n(\lambda)=\sum_{\nu=1}^{n}\log\left(1+\lambda\frac{p_\nu}{P_\nu}\right)$$

$$=\lambda\sum_{\nu=1}^{n}\frac{p_\nu}{P_\nu}+\sum_{\nu=1}^{n}O\left(\frac{p_\nu^2}{P_\nu^2}\right)\qquad(n\to\infty)$$

$$=\lambda\sum_{\nu=1}^{n}\frac{p_\nu}{P_\nu}+A(\lambda;\,p)+o(1)$$

$$=\lambda\log P_n+A_1(\lambda;\,p)+o(1).$$

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