PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 24 (38), 1978, pp. 125—132

ON OSTROWSKI'S FUNDAMENTAL EXISTENCE THEOREM

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(Received October 6. 1978)

Abstract. We give another proof of Ostrovski's fundamental existence theorem for Newton-Raphson method in the real case. The idea of the proof recalls to some of Cauchy's thoughts [1] and is used in the proof of Theorem 2 which, in the case on polynomials, gives sufficient conditions for convergence of Newton-Raphson method in terms of initial point only.

1. Ostrovski's fundamental existence theorem in the real case is the following [2]:

Theorem 1. *Let $f(x)$ be a real function of the real variable $x, f(x_0)$. $f'(x_0)\neq 0$ and put $h_0=-f(x_0)/f'(x_0)$, $x_1=x_0+h_0$. Consider the interval $J_0=[x_0,\ x_0+2h_0]$ and assume that $f'(x)$ exists in J_0, that $\max\limits_{J_0}|f''(x)|=M$ and*

$$2M|h_0|\leqslant|f'(x_0)|.$$

Let us form, starting with x_0, the sequence (x_i) by recurrence formula

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)} \tag{1}$$

Then all x_i lie in J_0 and we have

$$x_i\to\zeta \quad (i\to\infty)$$

where ζ is the only zero of f in J_0. Unless $\zeta=x_0+2h_0$ ζ is a simple zero. Further, we have the relations:

a) $$\frac{|x_{i+1}-x_i|}{|x_i-x_{i-1}|^2}\leqslant\frac{M}{2|f'(x_i)|} \qquad (i=1,2,\ldots)$$

b) $$|\zeta-x_{i+1}|\leqslant\frac{M}{2|f'(x_i)|}|x_i-x_{i-1}|^2 \qquad (i=1,2,\ldots)$$

Proof. First ew introduce some definitions:

$$h_i \stackrel{\text{def}}{=} \frac{-f(x_i)}{f'(x_i)}. \text{ i.e. } \qquad h_i = x_{i+1} - x_i$$

$$J_i \stackrel{\text{def}}{=} \{x \mid |x - x_{i+1}| \leqslant |x_{i+1} - x_i|\},$$

U_i $(i=1, 2, \ldots)$ is, by definition, the conjunction of the following formulae

(i) $$f'(x_i) \neq 0$$

(ii) $$|h_i| \leqslant 1/2 \, |h_{i-1}|$$

(iii) $$2M \, |h_i| \leqslant |f'(x_i)|.$$

For $i=0$ we define U_0 as $f'(x) \neq 0 \wedge 2M \; |h_0| \leqslant |f'(x_0)|$. The main idea is to prove U_i for all $i=0, 1, \ldots$. We prove this by complete induction on i. For $i=0$ the statement is true by assumptions of Theorem 1. Let $U_0 \wedge \cdots \wedge U_i$ be the induction hypothesis. Then by definition of $J_0, \ldots, J_i$ we have $J_0 \supseteq J_1 \supseteq \cdots \supseteq J_i$ and, as $f''(x)$ exists in $J_{0'}$, $\max\limits_{J_i} |f''(x)| \leqslant M$. Hence, since $x_1 + \theta(x_{i+1} - x_i) \in J_i$ if $0 < \theta < 1$, it follows that

$$\begin{aligned} |f'(x_{i+1}) - f'(x_i)| &= |f''(x_i + \theta(x_{i+1} - x_i))(x_{i+1} - x_i)| \\ &\leqslant M \, |h_i| \\ &\leqslant 1/2 \, |f'(x_i)|. \end{aligned}$$

Thus

(2) $$|f'(x_{i+1}) - f'(x_i)| \leqslant 1/2 \, |f'(x_i)|.$$

Therefore we easily obtain the part (i) of U_i, i.e.

(3) $$f'(x_{i+1}) \neq 0.$$

By identity

$$f'(x_{i+1}) = f'(x_i) + f'(x_{i+1}) - f'(x_i)$$

we have

$$\begin{aligned} |f'(x_{i+1})| &\geqslant |f'(x_i)| - |f(x_{i+1}) - f'(x_i)| \\ &\geqslant |f'(x_i)| - 1/2 \, |f'(x_i)| \\ &= 1/2 \, |f'(x_i)|. \end{aligned}$$

Thus

(4) $$|f'(x_{i+1})| \geqslant 1/2 \, |f'(x_i)|.$$

Using the Taylor polynomial:

$$f(x_{i+1}) = f(x_i) + \frac{f'(x_i)}{1!}(x_{i+1} - x_i) + \frac{f''(x_i + \theta(x_{i+1} - x_i))}{2!}(x_{i+1} - x_i)^2 \qquad (0 < \theta < 1)$$

and the fact that

$$(x_{i+1}-x_i)f'(x_i)+f(x_i)=0$$

we get

$$f(x_{i+1})=\frac{f''(x_i+\theta(x_{i+1}-x_i))}{2!}(x_{i+1}-x_i)^2$$

and therefore

(5) $$|f(x_{i+1})|\leqslant\frac{M}{2}|h_i|^2.$$

From (4) and (5) it follows

(6) $$|h_{i+1}|\leqslant\frac{M|h_i|^2}{|f'(x_i)|}$$

that is

(7) $$\frac{|h_{i+1}|}{|h_i|}\leqslant\frac{M|h_i|}{|f'(x_i)|}.$$

Using the hypothesis U_i we obtain

$$\frac{|h_{i+1}|}{|h_i|}\leqslant 1/2.$$

The last relation is equivalent to the part (ii) of U_{i+1}. To prove the part (iii) we use (4) and (6):

$$\frac{2M|h_{i+1}|}{|f'(x_{i+1})|}\leqslant\frac{2M\cdot M\cdot|h_i|^2}{1/2|f'(x_i)|\cdot|f'(x_i)|}$$
$$=\left[\frac{2M|h_i|}{|f'(x_i)|}\right]^2$$
$$\leqslant 1\ (By\ 2M|h_i|\leqslant|f'(x_i)|)$$

The inequality

$$\frac{2M|h_{i+1}|}{|f'(x_{i+1})|}\leqslant 1$$

is equivalent to the part (iii) of U_{i+1}. Thus U_i is true for all $i=0, 1, \dots$. The remaining part of the proof is as in [2].

2. In the case $f(x)$ is a real polynomial, we have the following existence theorem.

Theorem 2. Let

$$f(x)=x^n+p_{n-1}x^{n+1}+\cdots+p_1x+p_0$$

be a real polynomial, $f(x_0)\cdot f'(x_0)\neq 0$ *and* (x_i) *the sequence determined by recurrence formula* (1). *Further let* J_i, h_i $(i=0, 1, \dots)$ *be defined as in the preceding paragraph. Assume that* x_0 *satisfies the condition*

$$M(x_0)|h_0|\leqslant q|f'(x_0)|\ (q=0{,}3465735903)$$

where $M(x) \stackrel{\text{def}}{=} \max\{|f'(x)|, |f''(x)|, \ldots, |f^{(n)}(x)|\}$ – x *is a fixed real number. Then all* x_i *lie in* J_0 *and we have*

$$x_i \to \zeta \qquad (i \to \infty)$$

where ζ *is the only real root in* J_0 *of* $f(x)=0$. *Unless* $\zeta = x_0 + 2h_0$, ζ *is a simple root. Further, we have the relations*

$$(\alpha) \qquad \frac{|x_{i+1}-x_i|}{|x_i-x_{i-1}|^2} \leqslant \frac{e^q-1-q}{q^2}\,\frac{M(x_i)}{2|f'(x_i)|} \qquad (i=1,\ 2,\ldots)$$

$$(\beta) \qquad |\zeta - x_{i+1}| \leqslant \frac{e^q-1-q-}{q^2}\,\frac{M(x_i)}{2|f'(x_i)|}\cdot|x_i-x_{i-1}|^2 \qquad (i=1,\ 2,\ldots)$$

where $(e^q-1-q):q^2 \approx 0{,}5631349592$.

P r o o f. By induction on i we prove U_i, where U_i $(i=1,\ 2,\ldots)$ is the conjunction of formulae

$$\text{(j)} \qquad f'(x_i) \neq 0$$

$$\text{(jj)} \qquad |h_i| \leqslant 1/2\,|h_{i-1}|$$

$$\text{(jjj)} \qquad M(x_i)\,|h_i| \leqslant q\,|f'(x_i)|$$

and, in particular, U_0 is defined as $f'(x_0) \neq 0 \wedge M(x_0)\,|h_0| \leqslant q\,|f'(x_0)|$. If $i=0$, U_0 is true by assumption of the theorem. Let U_i be the induction hypothesis. $f(x)$ being a polynomial satisfies the identity

$$f'(x_{i+1}) = f'(x_i) + \frac{f''(x_i)}{1!}(x_{i+1}-x_i) + \cdots + \frac{f^{(n)}(x_i)}{(n-1)!}(x_{i+1}-x_1)^{n-1}$$

Therefore we obtain

$$|f'(x_{i+1}) - f'(x_i)| \leqslant \frac{|f''(x_i)|}{1!}|x_{i+1}-x_i| + \cdots + \frac{|f^{(n)}(x_i)|}{(n-1)!}|x_{i+1}-x_i|^{n-1}$$

and further

$$|f(x_{i+1}) - f'(x_i)| \leqslant M(x_i)\left[\frac{|h_i|}{1!} + \cdots + \frac{|h_i|^{n-1}}{(n-1)!}\right]$$

$$\leqslant M(x_i)\left[\frac{|f'(x_i)|}{1!\,M(x_i)}q + \cdots + \frac{|f'(x_i)|^{n-1}}{(n-1)!\,M(x_i)^{n-1}}q^{n-1}\right]$$

$$= |f(x_i)|\left[\frac{q}{1!} + \frac{|f'(x_i)|}{M(x_i)}\,\frac{q^2}{2!} + \cdots + \frac{|f'(x_i)|^{n-2}}{M(x_i)^{n-2}}\,\frac{q^{n-1}}{(n-1)!}\right]$$

$$\leqslant |f'(x)|\left[\frac{q}{1!} + \frac{q^2}{2!} + \cdots + \frac{q^{n-1}}{(n-1)!}\right]$$

(Since $|f(x_i)| \leqslant M(x_i)$)

$$\leqslant |f'(x)|\,(e^q-1).$$

Thus it has been proved that

(9) $$|f'(x_{i+1})-f'(x_i)|\leqslant(e^q-1)|f'(x_i)|$$

Since $e^q<2$ we conclude

(10) $$|f'(x_{i+1})-f'(x_i)|<|f'(x_i)|$$

and therefore

(11) $$f'(x_{i+1})\neq 0$$

We prove now the inequality

(12) $$|h_{i+1}|<1/2\,|h_i|$$

By identity

$$f'(x_{i+1})=f'(x_i)+f'(x_{i+1})-f'(x_i)$$

and the inequality (11) we easily obtain

(13) $$|f'(x_{i+1})|\geqslant(2-e^q)|f'(x_i)|$$

Further, by identity

$$f(x_{i+1})=f(x_i)+\frac{f'(x_i)}{1!}(x_{i+1}-x_i)+\cdots+\frac{f^{(n)}(x_i)}{n!}(x_{i+1}-x_i)^n$$

and the definition of the sequence (x_i), we have

$$|f(x_{i+1})|\leqslant|f''(x_i)|\frac{|h_i|^2}{2!}+\cdots+|f^{(n)}(x_i)|\frac{|h|^n}{n!}$$

$$\leqslant M(x_i)\left[\frac{|h_i|^2}{2!}+\cdots+\frac{|h_i|^n}{n!}\right]$$

$$=M(x_i)|h_i|^2\left[\frac{1}{2!}+\frac{|h_i|}{3!}+\cdots+\frac{|h_i|^{n-2}}{n!}\right]$$

$$\leqslant M(x_i)|h_i|^2\left[\frac{1}{2!}+\frac{q}{3!}+\cdots+\frac{q^{n-2}}{n!}\right]$$

(By U_i and the definition of $M(x_i)$)

$$=\frac{M(x_i)|h|^2}{q^2}\left[\frac{q^2}{2!}+\frac{q^3}{3!}+\cdots+\frac{q^n}{n!}\right]$$

$$\leqslant\frac{M(x_i)|h_i|^2}{q^2}\cdot\frac{e^q-1-q}{q^2}$$

Thus we have proved

(14) $$|f(x_{i+1})|\leqslant\frac{e^q-1-q}{q^2}M(x_i)|h_i|^2$$

Using (12) and (13) we obtain

$$|h_{i+1}| \leqslant \frac{e^q - 1 - q}{q^2(2 - e^2)} \cdot \frac{M(x_i)|h_i|^2}{|f'(x_i)|} \tag{15}$$

and therefore

$$\begin{aligned}\frac{|h_{i+1}|}{|h_i|} &\leqslant \frac{e^q - 1 - q}{q^2(2 - e^2)} \cdot \frac{M(x_i)|h_i|}{|f'(x_i)|}\\ &\leqslant \frac{e^q - 1 - q}{q(2 - e^q)}, \text{ by the part (jj) of } U_i\\ &\leqslant 1/2, \text{ what can be easily proved.}\end{aligned}$$

The inequality $\frac{|h_{i+1}|}{|h_i|} \leqslant 1/2$ is just (12).

We use now the identity

$$f^{(m)}(x_{i+1}) = f^{(m)}(x_i) + \frac{f^{(m+1)}(x_i)}{1!}(x_{i+1} - x_i) + \cdots + \frac{f^{(n)}(x_i)}{(n-m)!}(x_{i+1} - x_i)^{n-m}$$

$$(m = 1, 2, \ldots, n)$$

to prove the inequality

$$M(x_{i+1}) \leqslant e^q M(x_i). \tag{16}$$

Namely, from the above identity we have

$$\begin{aligned}|f^{(m)}(x_{i+1})| &\leqslant M(x_i)\left[1 + \frac{|h_i|}{1!} + \cdots + \frac{|h_i|^{n-m}}{(n-m)!}\right]\\ &\leqslant M(x_i)\left[1 + \frac{q}{1!} + \cdots + \frac{q^{n-m}}{(n-m)!}\right]\\ &\leqslant e^q M(x_i)\end{aligned}$$

therefrom we obtain (16).

Let us now prove the inequality

$$M(x_{i+1})|h_{i+1}| \leqslant q|f'(x_{i+1})| \tag{17}$$

Using (12) (14) and (15) we have

$$\frac{|h_{i+1}|M(x_{i+1})}{q|f'(x_{i+1})|} \leqslant \left[\frac{M(x_i)|h_i|}{|f'(x_i)|q}\right]^2 \cdot \frac{(e^q - 1 - q)e^q}{(2 - e^q)^2 q}$$

The expression in bracket is less or equal to 1 by induction hypothesis. Further, it is easy to see that the inequality

$$\frac{(e^q - 1 - q)e^q}{(2 - e^q)^2 q} \leqslant 1$$

holds and therefore the inequality

$$\frac{|h_{i+1}|\, M(x_{i+1})}{q\,|f'(x_{i+1})|} \leqslant 1,$$

which is equivalent to (17), holds as well.

The formulae (10), (15) and (17) form the assertion U_{i+1}. Therefore, we have a sequence of intervals

$$J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$$

with the radius of J_{i+1} at the most equal to one-half the radius of J_i. Such a sequence converges to a point ζ which lies in J_0. It can easily be proved that ζ is a zero of $f(x)$ and, unless $\zeta = x_0 + 2h_0$, ζ is a simple zero. The last assertion follows from the inequality

(18) $$|f'(x) - f'(x_0)| < |f'(x_0)| \qquad (\text{for } x \text{ inside } J_0)$$

that can be proved in the same way as (9). Namely, for x inside J_0 we have $|x - x_0| < 2|h_0|$ and therefore:

$$\begin{aligned}
|f'(x) - f(x_0)| &< M(x_0)\left[\frac{2|h_0|}{1!} + \cdots + \frac{2^{n-1}|h_0|^{n-1}}{(n-1)!}\right] \\
&\leqslant M(x_0)\left[\frac{2|f'(x_0|}{1!\,M(x_0)}q + \cdots + \frac{2^{n-1}|f'(x_0)|^{n-1}}{(n-1)!\,M(x_0)^{n-1}}q^{n-1}\right] \\
&= |f'(x_0)|\left[\frac{2q}{1!} + \frac{|f'(x_0)|}{M(x_0)}\frac{(2q)^2}{2!} + \cdots + \frac{|f'(x_0)|^{n-2}}{M(x_0)^{n-2}}\frac{(2q)^{n-1}}{(n-1)!}\right] \\
&\leqslant |f'(x_0)|\left[\frac{2q}{1!} + \frac{(2q)^2}{2!} + \cdots + \frac{(2q)^{n-1}}{(n-1)!}\right] \\
&\leqslant |f'(x_0)|\,(e^{2q} - 1)^1 \\
&\leqslant |f'(x_0)|
\end{aligned}$$

We prove now that ζ is the only zero in J_0. While by (18) $f'(x)$ does not vanish in J_0, whence $f(x)$ is strictly monotonically increasing in J_0 and thus has only one zero. In the case $\zeta = x_0 + 2h_0$ it can be shown easily that $f(x)$ is a quadratic polynomial with ζ as a double zero.

The assertions (α) and (β) of Theorem 2 can now be easily deduced. Namely, (α) is equivalent to

(19) $$|h_i| \leqslant \frac{e^q - 1 - q}{q^2}\,\frac{M(x_{i-1})\,|h_{i-1}|^2}{|f'(x_i)|}$$

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[1] The number 0,3465735903 has been determined so that inequality $e^{2q} - 1 < 1$ holds.

We use (13) and the fact that U_i is true for all $i=0, 1, \ldots$. We have therefore

$$|f(x_i)| \leqslant \frac{e^q - 1 - q}{q^2} M(x_{i-1}) \qquad |h_{i-1}|^2$$

and by definition of h_i we obtain (19).

The assertion (β) follows from (α) and the fact that ζ in J_i, i.e. in an interval with the center x_{i+1} and radius $|h_i|$. Thus $|\zeta - x_{i+1}| \leqslant |h_i|$. Our theorem is completely proved.

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