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FACTORIZATION OF QUASIHYPONORMAL OPERATORS

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Throughout our discussion, by an operator, we shall mean a bounded linear transformation on a complex Hilbert space H. If T is an operator, then we write $R(T)$ and $N(T)$ to denote the range and the null space of T. T is called quasinormal if $T(T^*T)=(T^*T)T$; hyponormal if $T^*T\geqslant TT^*$, and quasihyponormal if $T^{*2}T^2\geqslant(T^*T)^2$ [4]. It is well known that {Quasinormal operators} $\subseteq$ {Hyponormal operators} $\subseteq$ {Quasihyponormal operators}.

In [2], Embry has studied operators T for which $R(T)\subseteq R(T^*)$. Taking hint from this study, we introduce operators T for which $R(T^*T)\subseteq R(T^{*2})$. Let D_1 be the collection of all such operators. By Douglas' theorem [1], it is clear that D_1 includes all quasihyponormal operators and for T in this class, there exists a unique operator C such that

(i) $T^*T=T^{*2}C$;

(ii) $\|C\|=\inf\{u:u\geqslant 0 \text{ and } (T^*T)^2\leqslant uT^{*2}T^2\}$;

(iii) $N(T)=N(C)$, and

(vi) $R(C)\subseteq\overline{R(T^2)}$.

As done in [2], we shall characterize quasihyponormal, normal and selfadjoint operators in terms of C. In what follows, T will be an operator of class D_1.

Our first result gives characterization for quasihyponormal operators.

Theorem 1. *An operator T is quasihyponormal if and only if C is a contraction.*

Proof. Suppose C is a contraction. Then $\|T^*Tx\|=\|C^*T^2x\|\leqslant\|T^2x\|$ for all x in H and hence T is quasihyponormal.

Conversely assume that T is quasihyponormal. Since $\|C^*T^2x\|=\|T^*Tx\|\leqslant\leqslant\|T^2x\|$ for all x in H, $\|C^*y\|\leqslant\|y\|$ for all y in $\overline{R(T^2)}$. Now, as noted earlier, $R(C)\subseteq\overline{R(T^2)}$ or $N(C^*)\supseteq\overline{R(T^2)}^{\perp}$, therefore $C^*x=0$ for all x in $\overline{R(T^2)}^{\perp}$. In consequence, $\|C^*x\|\leqslant\|x\|$ for all x in H, that is, C is a contraction.

To characterize normal operators, we shall use the following result.

Theorem 2. *If T is a quasinormal operator, then C is a quasinormal partial isometry with* $\overline{R(T^2)} = R(C)$.

Proof. First we show C to be a partial isometry. Since T is quasinormal, $\| C^* T^2 x \| = \| T^* T x \| = \| T^2 x \|$ for all x in H. This shows that C^* is an isometry on $\overline{R(T^2)}$. But $\overline{R(T^2)} \supseteq R(C) = N(C^*)^{\perp}$. Therefore C^* is a partial isometry and hence $\overline{R(T^2)} = R(C)$ and C is a partial isometry.

To prove that C is quasinormal, we note that in view of the quasinormality of T, $N(T) \subseteq N(T^*)$. Since $N(C) \subseteq N(T)$ and $N(T^*) \subseteq N(C^*)$, we have $N(C) \subseteq N(C^*)$ or $R(C^*) \supseteq R(C)$. But C^*C and CC^* are projections on $R(C^*)$ and $R(C)$ respectively. Therefore C is hyponormal and hence quasinormal (see [3, Problem 161]).

Now we characterize normal operators in

Theorem 3. *T is normal if and only if C is a normal partial isometry with* $\overline{R(T)} = R(C)$.

Proof. Suppose T is normal. Then by Theorem 2, C^* is a partial isometry and $\overline{R(T^2)} = R(C)$. Since $\overline{R(T)} = \overline{R(T^2)}$, we have $\overline{R(T)} = R(C)$. This along with the relation $N(T) = N(C)$ yields $R(C) = \overline{R(T)} = N(T^*)^{\perp} = N(T)^{\perp} = N(C)^{\perp} = R(C^*)$. Since C^*C is the projection on $R(C^*)$ while CC^* is the projection on $R(C)$, we conclude that $C^*C = CC^*$.

On the other hand, if C is a normal partial isometry with $\overline{R(T)} = R(C)$, then $N(T) = N(C) = N(C^*) = R(C)^{\perp} = \overline{R(T)}^{\perp} = N(T^*)$. Therefore $\| T^* x \| = \| T x \|$ for x in $\overline{R(T)}^{\perp}$. Also $\| T^* T x \| = \| C^* T^2 x \| = \| T^2 x \|$ as C^* is a partial isometry on $\overline{R(T)}$. This shows that $\| T^* x \| = \| T x \|$ for all x in H or T is normal.

Lastly we characterize selfadjoint operators in Theorem 4.

Theorem 4. *T is selfadjoint iff C is the projection on* $\overline{R(T)}$.

Proof. Suppose T is selfadjoint. Since $T^* T = T^{*2} C$, $T^2 = T^2 C$, and hence $T = TC$. This shows that $C^* = I$ on $\overline{R(T)}$. Moreover as $N(C^*) \supseteq N(T^*)$, $C* = 0$ on $\overline{R(T)}^{\perp}$. Therefore we conclude that C is the projection on $\overline{R(T)}$.

Now let us assume that C is the projection on $\overline{R(T)}$. Then $T^* T = T^{*2} C = CT^2 = T^2$; thus $T^* y = T y$ for all y in $\overline{R(T)}$. To complete the proof, we show that $T^* x = T x$ for all x in $\overline{R(T)}^{\perp}$. To this end, note that $N(T) = N(C) = N(C^*)$ and $R(C) \subseteq \overline{R(T^2)} \subseteq \overline{R(T)}$ will give $N(T^*) \subseteq N(T)$. Consequently $T^* x = T x$ for all x in $\overline{R(T)}^{\perp}$.

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