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A MULTIPLICATION OF M-RELATIONS

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Let $E_1, \ldots, E_m$ be arbitrary sets. An m-relation between elements of the sets $E_1, \ldots, E_m$ is every subset R of Cartesian product $E_1 \times \cdots \times E_m$. The elements $x_1 \in E_1, \ldots, x_m \in E_m$ are said to be in the relation R in given order if $(x_1, \ldots, x_m) \in R$. The elements $x_1, \ldots, x_m$ are not in the relation R in given order if $(x_1, \ldots, x_m) \notin R$. If $E_1 = \cdots = E_m = E$, then $R \subset E^n$ and we say that R is an m-relation in the set E. The set E is the basis of the relation R.

The relation $R^{-1} \subset E_m \times E_{m-1} \times \cdots \times E_1$ is the inverse relation of the relation R if the following

$$(x_1, \ldots, x_m) \in R \Leftrightarrow (x_m, \ldots, x_1) \in R^{-1}$$

holds for $x_1 \in E_1, \ldots, x_m \in E_m$.

Operations on sets, such as union, intersection, difference, Cartesian product etc., can be realised on m-relations, too.

In addition, there exist also other operations on m-relations (see, for example, [1]).

One of them is De Morgan's operation which associates with an m-relation $R \subset E_1 \times \cdots \times E_{m-1} \times E_m$ and n-relation $S \subset E_m \times E_{m+1} \times \cdots \times E_{m+n-1}$ an $(m+n-2)$-relation $R*S \subset E_1 \times \cdots \times E_{m-1} \times E_{m+1} \times \cdots \times E_{m+n-1}$ such that

$$(x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_{m+n-1}) \in R*S \Leftrightarrow (\exists x_m \in E_m)$$

$$((x_1, \ldots, x_{m-1}, x_m) \in R \wedge (x_m, \ldots, x_{m+n-1}) \in S)$$

holds.

De Morgan's operation ([1]) is associative, that is

$$(R*S)*T = R*(S*T).$$

Furthermore, we have

$$(R*S)^{-1} = S^{-1}*R^{-1}.$$

If R and S are binary relations such that $R \subset E \times F$, $S \subset F \times G$, then the operation $*$ represents the well-known multiplication or composition of binary relations.

In this paper we shall consider a multiplication (or composition) of m-relations where the product is an m-relation, too.

Multiplying of two m-tuples will be defined in the following way:

Definition 1. If $m=2n$, let be

$$(x_{2n}, \ldots, x_{n+1}, x_{2n+1}, \ldots, x_{3n})\circ(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n})= \tag{1}$$
$$=(x_1, \ldots, x_n, x_{2n+1}, \ldots, x_{3n});$$

if $m=2n+1$, let be

$$(x_{2n+1}, \ldots, x_{n+1}, x_{2n+2}, \ldots, x_{3n+1})\circ(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n+1})= \tag{2}$$
$$=(x_1, \ldots, x_{n+1}, x_{2n+2}, \ldots, x_{3n+1}).$$

If $n=1$, the equality (1) gives the well-known multiplication of ordered pairs: $(x_2, x_3)\circ(x_1, x_2)=(x_1, x_3)$. The ordered triples will be multiplicated, by equality (2), in the following way: $(x_3, x_2, x_4)\circ(x_1, x_2, x_3)=(x_1, x_2, x_4)$. The ordered 4-tuples will be multiplicated, by equality (1), as follows:

$$(x_4, x_3, x_5, x_6)\circ(x_1, x_2, x_3, x_4)=(x_1\ x_2\ x_5\ x_6).$$

Definition 2. If $R\subset E_1\times E_2\times\cdots\times E_n\times E_{n+1}\times\cdots\times E_{2n}$ and $S\subset E_{2n}\times\cdots\times E_{n+1}\times E_{2n+1}\times\cdots\times E_{3n}$ are $2n$-relations, then the product or composition of relations R and S is a $2n$-relation $S\circ R\subset E_1\times E_2\times\cdots\times E_n\times E_{2n+1}\times\cdots\times E_{3n}$, such that

$$(x_1, x_2, \ldots, x_n, x_{2n+1}, \ldots, x_{3n})\in S\circ R \Leftrightarrow$$
$$(\exists\, x_{n+1}\in E_{n+1})\ (\exists\, x_{n+2}\in E_{n+2})\cdots(\exists\, x_{2n}\in E_{2n})$$
$$((x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{2n})\in R\wedge(x_{2n}, \ldots, x_{n+1}, x_{2n+1}, \ldots, x_{3n})\in S)$$

holds.

If $R\subset E_1\times E_2\times\cdots\times E_n\times E_{n+1}\times\cdots\times E_{2n+1}$ and $S\subset E_{2n+1}\times\cdots\times E_{n+1}\times E_{2n+2}\times\cdots\times E_{3n+1}$ are $2n+1$-relations, then the product of relations R and S is a $2n+1$-relation $S\circ R\subset E_1\times E_2\times\cdots\times E_{2n+2}\times\cdots\times E_{3n+1}$, such that

$$(x_1, x_2, \ldots, x_n, x_{n+1}, x_{2n+2}, \ldots, x_{3n+1})\in S\circ R \Leftrightarrow$$
$$(\exists\, x_{n+2}\in E_{n+2})\ (\exists\, x_{n+3}\in E_{n+3})\cdots(\exists\, x_{2n+1}\in E_{2n+1})$$
$$((x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{2n+1})\in R\wedge(x_{2n+1}, \ldots, x_{n+1}, x_{2n+2}, \ldots, x_{3n+1})\in S)$$

holds.

P r o p o s i t i o n 1. *The above defined multiplication of m-relations is associative*:

$$(T\circ S)\circ R = T\circ (S\circ R).$$

P r o o f. Let R, S and T be $2n$ – relations. Then

$$(x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{2n}) \in (T\circ S)\circ R \Leftrightarrow$$
$$(\exists y_1, \dots, y_n)\ ((x_1, x_2, \dots, x_n, y_1, \dots, y_n) \in R \wedge$$
$$(y_n, \dots, y_1, x_{n+1}, \dots, x_{2n}) \in T\circ S) \Leftrightarrow$$
$$(\exists y_1, \dots, y_n)\ ((x_1, x_2, \dots, x_n, y_1, \dots, x_n) \in R \wedge$$
$$(\exists z_1, \dots, z_n)\ ((y_n, \dots, y_1, z_1, \dots, z_n) \in S \wedge$$
$$(z_n, \dots, z_1, x_{n+1}, \dots, x_{2n}) \in T)) \Leftrightarrow$$
$$(\exists y_1, \dots, y_n)\ (\exists z_1, \dots, z_n)\ ((x_1, x_2, \dots, x_n, y_1, y_n) \in R \wedge$$
$$((y_n, \dots, y_1, z_1, \dots, z_n) \in S \wedge$$
$$(z_n, \dots z_1, x_{n+1}, \dots, x_{2n}) \in T)) \Leftrightarrow$$
$$(\exists z_1, \dots, z_n)\ ((\exists y_1, \dots, y_n)\ ((x_1, x_2, \dots, x_n, y_1, \dots, y_n) \in R \wedge$$
$$(y_n, \dots, y_1, z_1, \dots, z_n) \in S) \wedge$$
$$(z_n, \dots, z_1, x_{n+1}, \dots, x_{2n}) \in T) \Leftrightarrow$$
$$(\exists z_1, \dots, z_n)\ ((x_1, x_2, \dots, x_n, z_1, \dots, z_n) \in S\circ R \wedge$$
$$(z_n, \dots, z_1, x_{n+1}, \dots, x_{2n}) \in T) \Leftrightarrow$$
$$(x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{2n}) \in T\circ (S\circ R).$$

If R, S and T are $2n+1$ – relation, the proof runs in a similar way.

P r o p o s i t i o n 2. *If R and S are m-relations, then the following*

$$(S\circ R)^{-1} = R^{-1}\circ S^{-1}$$

holds.

P r o o f. Let R and S be $2n$ – relation:

$$(x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}) \in (S\circ R)^{-1} \Leftrightarrow (x_{2n}, \dots, x_{n+1}, x_n, \dots, x_1) \in S\circ R \Leftrightarrow$$
$$(\exists y_1, \dots, y_n)\ ((x_{2n}, \dots, x_{n+1}, y_1, \dots, y_n) \in R \wedge (y_n, \dots, y_1, x_{n+1}, \dots, x_1) \in S) \Leftrightarrow$$
$$(\exists y_1, \dots, y_n)\ ((x_1, \dots, x_n, y_1, \dots, y_n) \in S^{-1} \wedge (y_n, \dots, y_1, x_{n+1}, \dots, x_{2n}) \in R^{-1})$$
$$\Leftrightarrow (x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}) \in R^{-1}\circ S^{-1}.$$

The proof is analogous in the case when R and S are $2n+1$ – relation.

D e f i n i t i o n 3. An m-relation R in E (hence, $R \subset E^n$) is said to be symmetric if for every m-tuple $(x_1, \ldots, x_m) \in R$ it follows that $(s(x_1), \ldots, s(x_m)) \in R$, s being an element of the set of all permutations of elements $x_1, \ldots, x_m$.

E x a m p l e. A 3-relation $R = \{(0,0,0), (0,0,1), (0,1,0), (0,1,2), (0,2,1), (1,0,0), (1,0,2), (1,2,0), (1,2,2), (2,0,1), (2,1,0), (2,1,2), (2,2,1), (2,2,2)\}$ (Fig. 1) is a symmetric relation.

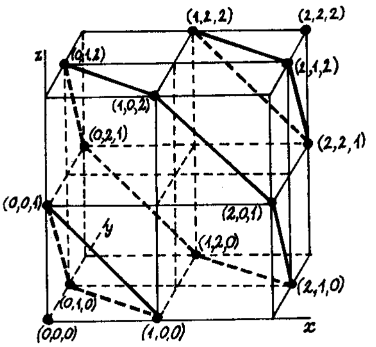

Fig. 1

It is seen that "symmetric" ordered triples are presented by the points that are vertices of a regular polyigon.

Remark that the Definition 3 for $n = 2$ presents the well-known definition of a symmetric binary relation.

P r o p o s i t i o n 3. *If R is a symmetric m-relation and R^{-1} its inverse relarion, then $R = R^{-1}$.*

P r o o f. Since R is a symmetric relation, we shall have for every ordered m-tuple $(x_1, \ldots, x_m)$:

$$(x_1, \ldots, x_m) \in R \Leftrightarrow (x_m, \ldots, x_1) \in R \Leftrightarrow (x_1, \ldots, x_m) \in R^{-1}.$$

Hence: $R = R^{-1}$.

The converse is not true.

D e f i n i t i o n 4. A relation R in E is said to be transitive, if together with all two ordered m-tuples, which can be multiplied by Definition 1, their product belongs to the relation R.

As a consequence of the above definition we can formulate the following proposition.

P r o p o s i t i o n 4. *An m-relation R in E is transitive if and only if*

$$R \circ R \subset R$$

Definition 5. An m-relation R in E is said to be reflexive if $(x, x, \dots, x) \in R$ for every $x \in E$.

Example. The 3-relation $R = \{(0,0,0), (1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2), (3,3,3), (3,3,4), (3,4,3), (3,4,4), (4,3,3), (4,3,4), (4,4,3), (4,4,4)\}$ in $E = \{0,1,2,3,4\}$ is reflexive, symmetric and transitive (Fig. 2).

Introduction of conception reflexive, symmetric and transitive m-relation gives us the possibility of defining an equivalence m-relation. The 3-relation from the above example is just an equivalence relation. Furthermore, it is seen that the given relation realises a partition of its basis $E = \{0,1,2,3,4\}$ into disjunct classes: $C_0 = \{0\}$, $C_1 = \{1,2\}$, $C_3 = \{3,4\}$. Such a conclusion is evidently possible in general for an m-relation.

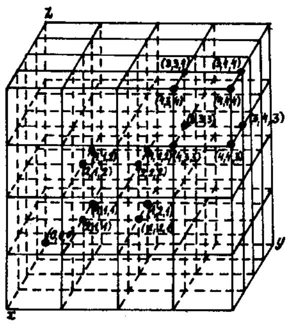

Fig. 2

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