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# SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS EQUIVALENT TO LINEAR DIFFERENTIAL EQUATIONS

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Summary. It is shown that the only nonlinear second order differential equations with solutions of the form $F(u, v)$, where $u$ and $v$ are independent solutions of a linear second order equation, are those which can be obtained from Pinney's equation $\eta''+p(x)\eta+c\eta^{-3}=0$ by simple transformations. This provides a simpler and a better description of those equations than the one given by Herbst and Gergen-Dressel.

**1.** Herbst [1] proved the following theorem.

Theorem 1. *If $u$ and $v$ are variable independent solutions with Wronskian $w$ of the linear equation*

$$Y''-w(x)^{-1}w'(x)Y'+q(x)Y=0 \tag{1}$$

*where $w$ and $q$ are given functions, then the equation*

$$y''-w^{-1}w'y'=f(y, y', w, q) \tag{2}$$

*has general solution*

$$y=F(u, v) \tag{3}$$

*if and only if*

$$f=-qZ(y)+A(y)(y')^2+w^2C(y) \tag{4}$$

*where $Z$, $A$, $C$ satisfy*

$$ZC'+(3-AZ)C=0, \qquad Z'-AZ=1. \tag{5}$$

The $F$ in (3) is any solution of the system

$$F_{uu}=A(F)F_u^2+v^2C(F), \qquad F_{vv}=A(F)F_v^2+u^2C(F),$$
$$F_{uv}=A(F)F_uF_v-uvC(F), \quad F_u=u^{-1}(Z(F)-vF_v).$$

"The main disadvantage of this result is the difficulty in determining $F$", comments Ames [2, p. 63].

The exact form of $F$ was obtained by Gergen and Dressel [3]. They proved the following theorem

Theorem 2. *Suppose that* $f(y, y', w, q) \in C'$ *on a domain* $R = \{(y, y', w, q) \mid m<y<M\}$, *that* $\xi(y) \equiv f(y, 0, 0, -1) \neq 0$, $m<y<M$, *and that* $F(u, v) \in C''$ *and* $m<F<M$ *on domain* $V$. *Suppose further that for arbitrary constants* $w \neq 0$ *and* $q$, *if* $u$ *and* $v$ *are solutions of* (1) *with Wronskian* $w$ *such that* $(u, v) \in V$ *for* $x$ *on an interval* $I$, *then* $y = F(u, v)$ *satisfies* (2) *on* $I$. *Under these conditions, if* $m<\eta<M$, *then* $F$ *can be written* $F = \Phi(\omega^{1/2})$, $(u, v) \in V$, *where* $\omega(u, v)$ *is a homogeneous polynomial of degree* 2, *positive in* $V$, *and* $\Phi$ *is the inverse of* $\varphi(y) = \exp \int_{\eta}^{y} 1/\xi(t)\, dt$, $m<y<M$.

The proof of theorem 2 given in [3] is based on the uniqueness properties of solutions of differential equations, and is rather long and involved.

The object of this note is to provide an elementary and straight-forward proof of the Gergen-Dressel result. This proof also gives a deeper insight into the nature of Herbst's equation (2) – (4)—(5).

**2.** It is readily verified that if $u$ and $v$ are linearly independent solutions of (1), then

$$z = (au^2 + buv + (1/4\, a)(4\, c + b^2)\, v^2)^{1/2}, \tag{6}$$

where $a \neq 0$ and $b$ are arbitrary constants, is the general solution of the equation

$$z'' - w^{-1} w' z' + qz = cw^2 z^{-3} \qquad (c = \text{const}). \tag{7}$$

This result with $b = 0$ can be found in Reid [4].

Putting

$$z = \exp \int dy/Z(y), \tag{8}$$

equation (7) is transformed into Herbst's equation (2) with (4) and (5). Hence, the general solution of (2)—(4)—(5) is $y = \Phi(z)$, where $z$ is given by (6) and $\Phi$ is the inverse of $\exp \int dy/Z(y)$.

This is a straight-forward proof of theorem 2.

**3.** Pinney [5] noted that the equation

$$\eta'' + p(x)\, \eta + c\, \eta^{-3} = 0 \qquad (c = \text{const}) \tag{9}$$

has the solution $y = (u^2 - v^2)^{1/2}$, where $u$ and $v$ are appropriately chosen solutions of the corresponding linear equation $\eta'' + p(x)\, \eta = 0$.

If we put $\eta = w^{-1/2} z$, then (9) transforms into (7), with $q = p + (3\, w'^2 - 2\, ww'')/4\, w^2$. Hence, Herbst's equation (2)—(4)—(5) can be obtained from Pinney's equation (9) by the substitution

$$\eta = w^{-1/2} \exp \int dy/Z(y). \tag{10}$$

This leads to the surprising conclusion that Pinney's equation (9), which initiated the research of Herbst and Gergen and Dressel, and which was generalized to (2)—(4)—(5), is actually the canonical equation for the class described by Herbst. In other words, the only second order equations with general solution $F(u, v)$, where $u$ and $v$ are independent solutions of a linear second order equation, are those which can be obtained from Pinney's equation (9) by a transformation of the form $\eta = f(x)\, g(y)$.

## REFERENCES

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