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ON SOME CLASSES OF LINEAR EQUATIONS

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In this paper we shall be concerned with the equation $P(L)\,x=0$, where P is a polynomial over **C**; $x\in V$, where V is a commutative algebra over **C** and L is a linear operator on V subjected to certain conditions.

1. The class $H(V)$

Definition 1. *Suppose that L is a linear operator on V which satisfies the condition*

$$(1)\qquad L(uv)=uLv \quad \textit{if and only if} \quad u\in\ker L.$$

The class of such operators will be denoted by $H(V)$.

Let P be an m-th degree polynomial over **C** and consider the equation

$$(2)\qquad P(L)\,x=0,$$

where $L\in H(V)$ and $x\in V$ is the unknown vector.

Theorem 1. *Suppose that $\lambda_1, \ldots, \lambda_n$ are roots of P and that they are, at the same time, characteristic values of L. If $v_1, \ldots, v_n$ are the corresponding characteristic vectors, then*

$$(3)\qquad x=\sum_{k=1}^{n} u_k v_k,$$

where $u_1, \ldots, u_n\in\ker L$ are arbitrary, is a solution of the equation (2).

Proof. Since, by hypothesis, $Lv_k=\lambda_k v_k$ $(k=1, \ldots, n)$, and since $L\in H(V)$, from (3) follows

$$Lx=\sum_{k=1}^{n} u_k Lv_k=\sum_{k=1}^{n} \lambda_k u_k v_k,$$

and further

$$L^{\nu} x = \sum_{k=1}^{n} \lambda_k^{\nu} u_k v_k \qquad (\nu = 1, \ldots, n).$$

Hence,

$$P(L)\, x = \sum_{k=1}^{n} P(\lambda_k)\, u_k v_k = 0,$$

since, by hypothesis, $P(\lambda_k) = 0$ for $k = 1, \ldots, n$.

Definition 2. *Let $x_1, \ldots, x_n \in V$. If there exist vectors $u_1, \ldots, u_n \in$ ker L, not all zero, such that*

$$\sum_{k=1}^{n} u_k x_k = 0, \tag{4}$$

we say that $x_1, \ldots, x_n$ are linearly dependent with respect to the kernel of L, or shortly (ker *L)-linearly dependent.*

On the other hand, if (4), *where $u_k \in$* ker *L, implies $u_1 = \cdots = u_n = 0$, we say that $x_1, \ldots, x_n$ are linearly independent with respect to the kernel of L, or* (ker *L)-linearly independent.*

Theorem 2. *Let $x, y \in V$ and let $L \in H(V)$. Those vectors are* (ker *L)-linearly dependent if and only if*

$$W(x, y) = \begin{vmatrix} x & y \\ Lx & Ly \end{vmatrix} = 0.$$

Proof. Suppose that x and y are (ker L)-linearly dependent. Then, we have, for instance, $y = ux$, where $u \in$ ker L. But then

$$W(x, ux) = \begin{vmatrix} x & ux \\ Lx & L(ux) \end{vmatrix} = \begin{vmatrix} x & ux \\ Lx & uLx \end{vmatrix} = 0.$$

Conversely, suppose that

$$\begin{vmatrix} x & y \\ Lx & Ly \end{vmatrix} = 0.$$

This implies that there exists $u \in V$ such that $y = ux$ and $Ly = uLx$. From those two equalities follows $L(ux) = uLx$, and hence, in viitue of (1), $u \in$ ker L.

2. The class $K(V)$

Definition 3. *Suppose that $L \in H(V)$ and suppose that the following condition is satisfied:*

If $x_1, \ldots, x_n$ are (ker *L)-linearly independent and if*

$$\sum_{k=1}^{n} L^{\nu}(u_k x_k) = \sum_{k=1}^{n} u_k L^{\nu} x_k \quad (\nu = 1, \ldots, n-1)$$

then $u_1, \ldots, u_n \in$ ker *L.*

We then say that $L \in K(V)$.

T h e o r e m 3. *Let* $x_1, \ldots, x_n \in V$ *and let* $L \in K(V)$. *Those vectors are* (ker L)-*linearly dependent if and only if*

$$(5) \qquad W(x_1, \ldots, x_n) \equiv \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ Lx_1 & Lx_2 & & Lx_n \\ \vdots & & & \\ L^{n-1}x_1 & L^{n-1}x_2 & & L^{n-1}x_n \end{vmatrix} = 0.$$

P r o o f. If $x_1, \ldots, x_n$ are (ker L)-linearly dependent, then, for instance,

$$x_r = \sum_{\substack{k=1 \\ k \neq r}}^{n} u_k x_k \quad \text{with} \quad u_k \in \ker L.$$

But then

$$L^\nu x_r = \sum_{\substack{k=1 \\ k \neq r}}^{n} u_k L^\nu x_k \qquad (\nu = 1, \ldots, n-1),$$

and clearly $W = 0$.

Conversely, suppose that (5) holds. Then one column, the r-th column say, can be expressed in terms of others, i.e.

$$(6) \qquad L^\nu x_r = \sum_{\substack{k=1 \\ k \neq r}}^{n} u_k L^\nu x_k \qquad (\nu = 0, \ldots, n-1).$$

This system implies

$$(7) \qquad \sum_{\substack{k=1 \\ k \neq r}}^{n} L^\nu (u_k x_k) = \sum_{\substack{k=1 \\ k \neq r}}^{n} u_k L^\nu x_k \qquad (\nu = 1, \ldots, n-1).$$

We now distinguish between two possibilities:

(i) Vectors $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n$ are (ker L)-linearly dependent. The theorem is in that case proved.

(ii) Vectors $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n$ are (ker L)-linearly independent. Then since $L \in K(V)$, the system (7) implies that $u_k \in \ker L$, and from (6) for $\nu = 0$ we see that the theorem is again proved.

Denote by S the space of all solutions of the equation

$$(8) \qquad P(L)\, x = 0$$

where $L \in H(V)$.

Clearly, if $v \in S$, then $uv \in S$ where $u \in \ker L$.

Theorem 4. *If* $L \in K(V)$, *then* $\dim S \leq \operatorname{dg} P$.

Proof. Let $m = \operatorname{dg} P$ and let $x_1, \ldots, x_{m+1}$ be arbitrary (but distinct) elements of S. Then

$$P(L)\,x_k = 0 \qquad (k = 1, \ldots, m+1). \tag{9}$$

Eliminating the coefficients of P between the $m+1$ equations (9) we get

$$\begin{vmatrix} x_1 & x_2 & \cdots & x_{m+1} \\ Lx_1 & Lx_2 & & Lx_{m+1} \\ \vdots & & & \\ L^m x_1 & L^m x_2 & & L^m x_{m+1} \end{vmatrix} = 0,$$

which implies, in view of Theorem 3, that the vectors $x_1, \ldots, x_{m+1}$ are (ker L)-linearly dependent, i.e. for some k we have

$$x_k = \sum_{\substack{\nu=1 \\ \nu \neq k}}^{m+1} u_\nu x_\nu$$

with $u_\nu \in \ker L$.

However, since $x_\nu \in S$, we have $u_\nu x_\nu \in S$, implying that $x_1, \ldots, x_{m+1}$ are linearly dependent vectors. Hence, $\dim S \leq m$.

Theorem 5. *If* $L \in K(V)$ *and if* $v_1, \ldots, v_m$ *are linearly independent solutions of the equation* (8), *then its general solution is*

$$x = \sum_{k=1}^{m} u_k v_k \tag{10}$$

where $u_k \in \ker L$ *are arbitrary.*

Proof. If S denotes, as before, the space of all solutions of the equation (8), then $v_1, \ldots, v_m$ are m linearly independent elements of S. In view of Theorem 4 $\dim S = m$, and $(v_1, \ldots, v_m)$ is a basis of S. But then $(u'_1 v_1, \ldots, u'_m v_m)$, where $(0 \neq)\, u'_\nu \in \ker L$, is also a basis of S. Hence, if $x \in S$, then

$$x = \sum_{k=1}^{m} \alpha_k u'_k v_k . \qquad (\alpha_k \text{ scalars}).$$

Denoting $\alpha_k u'_k$ by u_k we arrive at (10).

Theorem 6. *Suppose that* $\lambda_1, \ldots, \lambda_m$ *are distinct roots of* P *and that they are, at the same time, characteristic values of* $L \in K(V)$. *If* $v_1, \ldots, v_m$ *are the corresponding characteristic vectors, then*

$$x = \sum_{k=1}^{m} u_k v_k,$$

where $u_1, \ldots, u_m \in \ker L$ *are arbitrary, is the general solution of the equation* (8).

P r o o f. If $\lambda_1, \ldots, \lambda_m$ are distinct characteristic values, then the corresponding characteristic vectors are linearly independent. Hence, this theorem follows directly from Theorem 5.

On the basis of Theorem 6, we see that the equation $P(L)x=0$ can, in certain cases, be replaced by two simpler equations $Lx=0$ and $Lx=\lambda x$ (λ arbitrary scalar).

3. The class $D(V)$

D e f i n i t i o n 4. *Let L be a linear operator on V and let $\alpha \in \ker L$ be fixed. If for all $u, v \in V$*

$$L(uv) = uLv + vLu + \alpha\, Lu\, Lv$$

we say that $L \in D_\alpha(V)$.

T h e o r e m 7. *If $L \in D_\alpha(V)$, then $L \in H(V)$.*

P r o o f. Trival.

T h e o r e m 8. *If $\alpha \neq 0$, then $L \in D_\alpha(V)$ if and only if $\alpha L \in D_1(V)$.*

P r o o f. If $\alpha \neq 0$, the equalities

$$L(uv) = uLv + vLu + \alpha\, Lu\, Lv \quad \text{and} \quad \alpha L(uv) = u\,\alpha Lv + v\,\alpha Lu + \alpha Lu\, \alpha Lv$$

are clearly equivalent.

R e m a r k. This theorem shows that D_0 and D_1 are the only interesting subclasses of D_α.

T h e o r e m 9. *If $L \in D_1(V)$ and $w \in V$, where w is not a solution of the equation $w + Lw = 0$, then $wL \in K(V)$.*

P r o o f. Put $A = wL$. Then

$$A(uv) = uAv + vAu + w^{-1} Au\, Av, \tag{11}$$

and $\ker A = \ker L$.

Suppose that $x_1, \ldots, x_n$ are ($\ker L$)-linearly independent and that

$$\sum_{k=1}^{n} A^\nu (u_k x_k) = \sum_{k=1}^{n} u_k A^\nu x_k \qquad (\nu = 1, \ldots, n-1). \tag{12}$$

For $\nu = 1$ we find

$$\sum_{k=1}^{n} A(u_k x_k) = \sum_{k=1}^{n} u_k A x_k, \tag{13}$$

while in virtue of (11) we have

$$\sum_{k=1}^{n} A(u_k x_k) = \sum_{k=1}^{n} u_k Ax_k + \sum_{k=1}^{n} x_k Au_k + \sum_{k=1}^{n} w^{-1} Au_k Ax_k .$$

Hence,

$$\sum_{k=1}^{n} (x_k + w^{-1} Ax_k) Au_k = 0,$$

or, equivalently,

$$\sum_{k=1}^{n} (x_k + Lx_k) Lu_k = 0.$$

Applying the operator A to (13) we find

$$\sum_{k=1}^{n} A^2 (u_k x_k) = \sum_{k=1}^{n} u_k A^2 x_k + \sum_{k=1}^{n} Au_k Ax_k + \sum_{k=1}^{n} w^{-1} Au_k A^2 x_k ,$$

which, together with (12) for $\nu = 2$ yields

$$\sum_{k=1}^{n} (Ax_k + w^{-1} A^2 x_k) Au_k = 0,$$

or, equivalently,

$$\sum_{k=1}^{n} \big(wLx_k + L(wLx_k)\big) Lu_k = 0,$$

i.e.

$$\sum_{k=1}^{n} (w + Lw) L (x_k + Lx_k) Lu_k = 0.$$

Repeating this procedure, we arrive at the following system:

$$\sum_{k=1}^{n} X_k Lu_k = 0$$

$$\sum_{k=1}^{n} L(WL)^{\nu} X_k Lu_k = 0 \qquad (\nu = 0, 1, \ldots, n-2),$$

where $X_k = x_k + Lx_k$ and $W = w + Lw$ ($\neq 0$).

Put

$$D = \begin{vmatrix} X_1 & X_2 & \cdots & X_n \\ LX_1 & LX_2 & & LX_n \\ L(WL) X_1 & L(WL) X_2 & & L(WL) X_n \\ \vdots & & & \\ L(WL)^{n-2} X_1 & L(WL)^{n-2} X_2 & & L(WL)^{n-2} X_n \end{vmatrix} .$$

After a finite number of elementary transformations we arrive at the equality

$$D=(I+L)^{n(n-1)/2}\,W\begin{vmatrix} X_1 & X_2 & \cdots & X_n \\ LX_1 & LX_2 & & LX_n \\ \vdots & & & \\ L^{n-1}X_1 & L^{n-1}X_2 & & L^{n-1}X_n \end{vmatrix},$$

where I is the identity operator.

However, $(I+L)^{n(n-1)/2}\,W=(I+L)^{(n-2)(n+1)/2}\,w\neq 0$. On the other hand, it is readily shown that if $x_1, \ldots, x_n$ are (ker L)-linearly independent, then $X_1, \ldots, X_n$, where $X_k=x_k+Lx_k$ are also (ker L)-linearly independent. Therefore, according to Theorem 3, $D\neq 0$. This means that $Lu_1=\cdots=Lu_n=0$, i.e. $u_1, \ldots, u_n\in\ker L=\ker A$, and the theorem is proved.

Theorem 10. *If $L\in D_\alpha(V)$, then $L\in K(V)$.*

Proof. If $\alpha=0$, the result is readily verified. If $\alpha\neq 0$, and $L\in D_\alpha(V)$, then $\alpha L\in D_1(V)$ and hence $\alpha L\in K(V)$ (we let w of Theorem 9 be the unity of V). However, from Definition 3 directly follows that $L\in K(V)$ if and only if $\alpha L\in K(V)$ $(\alpha\neq 0)$.

4. Some particular cases

The above theorems unify some known results for equations which contain differential and difference operators. In order to illustrate this point, we consider some second order equatinos.

Let the polynomial $P(t)=t^2+pt+q\;(p,\ q\in\mathbf{R})$ have two distinct real roots λ and μ. The following equations

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0, \tag{14}$$

$$\frac{d^2y}{dx^2}+\frac{p+f'(x)}{f(x)}\frac{dy}{dx}+\frac{q}{f(x)^2}y=0, \tag{15}$$

$$f^2\frac{\partial^2u}{\partial x^2}+2fg\frac{\partial^2 y}{\partial x\,\partial y}+g^2\frac{\partial^2u}{\partial y^2}+(ff_x+gf_y+pf)\frac{\partial u}{\partial x}+ (fg_x+gg_y+pg)\frac{\partial u}{\partial y}+qu=0, \tag{16}$$

$$y(x+2)+(p-2)\,y(x+1)+(1+q-p)\,y(x)=0, \tag{17}$$

$$\Delta^2y(x)+\frac{p+\Delta f(x)}{f(x+1)}\Delta y(x)+\frac{q}{f(x)f(x+1)}y(x)=0, \tag{18}$$

where $\Delta y(x) = y(x+1) - y(x)$, are special cases of the equation

$$L^2 v + pLv + qv = 0.$$

For each of the above equations we specify the operator, its kernel, and the characteristic vectors v_λ, v_μ which correspond to the characteristic values λ and μ.

Eq. (14): $L = \dfrac{d}{dx}$; $\ker L = \mathbf{R}$; $v_\lambda = e^{\lambda x}$, $v_\mu = e^{\mu x}$

Eq. (15): $L = f(x) \dfrac{d}{dx}$; $\ker L = \mathbf{R}$; $v_\lambda = \exp \displaystyle\int \frac{\lambda\, dx}{f(x)}$, $v_\mu = \exp \displaystyle\int \frac{\mu\, dx}{f(x)}$

Eq. (16): $L = f(x, y) \dfrac{\partial}{\partial x} + g(x, y) \dfrac{\partial}{\partial y}$; $\ker L = \{F(\omega(x, y)) \mid F$ is a differentiable function and $\omega(x, y) = C$ is the general solution of $y' = f(x, y)/g(x, y)\}$; v_λ is any solution of $fv_x + gv_y = \lambda v$, v_μ is any solution of $fv_x + gv_y = \mu v$

Eq. (17): $L = \Delta$; $\ker L = P =$ the set of all periodic functions with period 1; $v_\lambda = (1+x)^\lambda$, $v_\mu = (1+x)^\mu$

Eq. (18): $L = f(x)\Delta$; $\ker L = P$; $v_\lambda = \displaystyle\prod_{\nu=0}^{x-1} \left(1 + \frac{\lambda}{f(\nu)}\right)$, $v_\mu = \displaystyle\prod_{\nu=0}^{x-1} \left(1 + \frac{\mu}{f(\nu)}\right)$.

Hence, applying Theorem 6 we see that the general solution of the equation (14) is:

$$y = C_1 e^{\lambda x} + C_2 e^{\mu x};$$

of equation (15) is:

$$y = C_1 \exp \int \frac{\lambda\, dx}{f(x)} + C_2 \exp \int \frac{\mu\, dx}{f(x)};$$

of equation (16) is:

$$u = A(x, y)\,\alpha(x, y) + B(x, y)\,\beta(x, y);$$

of equation (17) is:

$$y = p_1(x)(1+x)^\lambda + p_2(x)(1+x)^\mu;$$

and of equation (18) is:

$$y = p_1(x) \prod_{\nu=0}^{x-1} \left(1 + \frac{\lambda}{f(\nu)}\right) + p_2(x) \prod_{\nu=0}^{x-1} \left(1 + \frac{\mu}{f(\nu)}\right),$$

where C_1, C_2 are arbitrary constants, $p_1(x)$ and $p_2(x)$ are arbitrary periodic functions with period 1, $A(x, y)$ and $B(x, y)$ are arbitrary solutions of the equation $fu_x + gu_y = 0$, $\alpha(x, y)$, $\beta(x, y)$ are particular solutions of the equations $fu_x + gu_y = \lambda u$, $fu_x + gu_y = \mu u$, respectively.

Remark. The above list of possible interpretations of L is by no means complete. As an other important interpretation of L one could take Kolosov's operator D, defined for complex differentiable functions $w(z) =$

$u(x, y)+iv(x, y)$ by $Dw=\frac{1}{2}(u_x-v_y+i(v_x+u_y))$, which leads to systems of (real) partial differential equations.

5. Concluding remark

The object of this paper was to show how general solutions of apparently different linear equations can be obtained by the same method. It seems that we have also arrived at a new class of solvable equations. Namely, the general solutions of equations (14), (15), (16) and (17) were already known, but we have not found in literature the general solution of the equation (18). Some special cases of that equation were treated. Recently Smentek (Zesz. nauk. Ug. Matematyka 1971 (1972), No. 1, pp. 121—168) solved the equation (18) with $f(x)=x$, and called it Euler's difference equation.

We mention one more point in connection with the equation (18). It is well known that the differential equation (15) is the most general equation that can be reduced to an equation with constant coefficients. One might consider the question whether the difference equation (18) (which is analogous to (15)) is the most general difference equation that can be reduced to a difference equation with constant coefficients.