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ON COMMUTING FAMILIES OF SUBNORMAL OPERATORS

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bstract. It is proved that if $A=(A_1, \ldots, A_n)$ is a commuting family of subnormal operars on a complex Hilbert space then $\sigma(A)$ is a joint spectral set for A.

Let H be a complex Hilbert space and let $A=(A_1, \ldots, A_n)$ be an tuple of commuting bounded operators on H. Let $\mathcal{U}$ be the double communt of the set $\{A_1, \ldots, A_n\}$ i.e. the set of all operators on H which comute with every operator that commutes with each of $A_1, \ldots, A_n$. Then $\mathcal{U}$ is commutative Banach algebra with identity containing the set $\{A_1, \ldots, A_n\}$. We shall need the following definitions in the sequel.

A point $\lambda=(\lambda_1, \ldots, \lambda_n)$ of $\mathscr{C}^n$ (the n-dimensional complex space) is in the joint spectrum $\sigma(A)$ of A if for all $B_1, \ldots, B_n$ in $\mathcal{U}$

$$\sum_{i=1}^{n} B_i(A_i-\lambda_i)\neq I$$

A complex vector $\lambda=(\lambda_1, \ldots, \lambda_n)$ of $\mathscr{C}^n$ is in the joint point spectrum $\sigma_p(A)$ of A if there exists x in H such that

$$A_i x=\lambda_i x, \qquad 1\leqslant i\leqslant n.$$

A point $\lambda=(\lambda_1, \ldots, \lambda_n)$ is in the joint approximate point spectrum $\sigma_{\Pi}(A)$ of A if there exists a sequence $\{x_k\}$ of unit vectors in H such that

$$\|(A_i-\lambda_i)x_k\|\to 0 \text{ as } k\to\infty, \ 1\leqslant i\leqslant n.$$

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The joint numerical range $W(A)$ of A is the set of all points $\lambda=(\lambda_1, \ldots, \lambda_n)$ of $\mathscr{C}^n$ such that for some x in H with $\|x\|=1$, $\lambda_i=<A_i x, x>$ for each i. Thus

$$W(A)=\{<Ax, x>=(<A_1 x, x>, \ldots, <A_n x, x>): x\in H \text{ with } \|x\|=1\}.$$

A closed subset X of $\mathscr{C}^n$ is a joint spectral set for A if

$$\|u(A)\|\leqslant\sup_{\lambda\in X}|u(\lambda)|=\|u\|_X,$$

for all rational functions u without singularities on X.

Let $H_1, H_2, \ldots, H_n$ be complex Hilbert spaces, I_k the identity operator and A_k an arbitrary operator on H_k, $1\leqslant k\leqslant n$. Consider the tensor product space $H_1\otimes H_2\otimes\cdots\otimes H_n$ and the operators T_k's defined on it by

$$T_1=A_1\otimes I_2\otimes\cdots\otimes I_n,$$

$$T_2=I_1\otimes A_2\otimes\cdots\otimes I_n,$$

and, in general

$$T_k=I_1\otimes I_2\otimes\cdots\otimes I_{k-1}\otimes A_k\otimes I_{k+1}\otimes\cdots\otimes I_n$$

Obviously T_k's are commuting operators. Dash [3] has shown that

$$W(T)=\Pi\, W(T_i)=\Pi\, W(A_i)$$

$$\sigma(T)=\Pi\,\sigma(T_i)=\Pi\,\sigma(A_i),$$

where $T=(T_1, \ldots, T_n)$.

In this paper, we are interested in taking $A=(A_1, \ldots, A_n)$ to be a commuting family of subnormal operators A_i's on H. It is not known if there exists a commuting family $B=(B_1, \ldots, B_n)$ where B_i is the minimal normal extension of A_i on a Hilbert space K containing H as a subspace, $1\leqslant i\leqslant n$. However, if the family A is doubly commutative, then there exists a commuting family B in which each B_i is a normal extension of A_i, and this is the extent of our knowledge in this direction [1].

Dash [5] has proved that if $A=(A_1, \ldots, A_n)$ is a commuting family of normal operators on a complex Hilbert space H then $\sigma(A)$ is a joint spectral set for A. We shall extend this result for a commuting family $A=(A_1, \ldots, A_n)$ of subnormal operators A_i for which there exists a commuting family $B=(B_1, \ldots, B_n)$ where B_i is the minimal normal extension of A_i.

We observe that $\sigma_p(A)\subset\sigma_p(B)$. One may conjecture that $\sigma(A)\subset\sigma(B)$. That this is false is shown by the following example:

If U is the unilateral shift and B is the bilateral shift and if we take $A_1=U\otimes I$, $A_2=I\otimes U$, $B_1=B\otimes I$ and $B_2=I\otimes B$, then $\sigma(A)=\bar{\Delta}^2$, the cartesian product of 2 closed unit discs and $\sigma(B)=\Gamma^2$, the cartesian product of 2 copies of the unit circle [2]. However, we can prove the following:

T h e o r e m 1. *If* $A=(A_1, \ldots, A_n)$ *is a commuting family of subnormal operators on* H *and* $B=(B_1, \ldots, B_n)$ *is a commuting family of their minimal normal extensions on a Hilbert space* $K \supset H$, *then* $\sigma(B) \subset \sigma(A)$.

P r o o f. It is sufficient to prove that $(0, \ldots, 0) \notin \sigma(A)$ then $(0, \ldots, 0) \notin \sigma(B)$. For simplicity, we shall prove the theorem for $n=2$. However, our method can be used to prove the general case also. Let $(0, 0) \notin \sigma(A)$, where $A=(A_1, A_2)$. Then $\sum_{i=1}^{2} T_i A_i$ is invertible for some T_i 's in the double commutant of (A_1, A_2). Thus $\sum_{i=1}^{2} \left(\frac{T_i}{\|T_1\|+\|T_2\|} A_i\right)$ is invertible. We normalize this operator so that $\left\| \left(\sum_{i=1}^{2} \frac{T_i}{\|T_1\|+\|T_2\|} A_i\right)^{-1} \right\| = 1$. Let $T_i' = \frac{T_i}{\|T_1\|+\|T_2\|}$ so that $\|T_i'\| \leqslant 1$. For $0<\varepsilon<1$, let

$$E=\bigcap_{i=1}^{2}\left\{f \in K : \|B_i^m f\| \leqslant \frac{\varepsilon^m}{2^m} \|f\|, \; m=1, 2, 3, \ldots\right\}.$$

By Theorem 1 [8, p. 66], E is a reducing subspace for B_1 and B_2. Let $f \in E$ and $g \in H$, then

$$|\langle f, g \rangle| = \left|\left\langle f, \left(\sum_{i=1}^{2} T_i' A_i\right)^m \left(\sum_{i=1}^{2} T_i' A_i\right)^{-m} g\right\rangle\right|$$

$$= \left|\left\langle f, \left(\sum_{i=1}^{2} A_i T_i'\right)^m \left(\sum_{i=1}^{2} T_i' A_i\right)^{-m} g\right\rangle\right|$$

$$= \left|\left\langle f, (A_1^m T_1'^m + {}^mC_1 A_1^{m-1} A_2 T_1'^{m-1} T_2' + \cdots + A_2^m T_2'^m) \left(\sum_{i=1}^{2} T_i' A_i\right)^{-m} g\right\rangle\right|$$

$$= \left|\left\langle f, (B_1^m T_i'^m + mC_1 B_1^{m-1} B_2 T_1'^{m-1} T_2' + \cdots + B_2^m T_2'^m) \left(\sum_{i=1}^{2} T_i' A_i\right)^{-m} g\right\rangle\right|$$

$$\leqslant \|B_1^{*m} f\| \, \|g\| + {}^mC_1 \|B_1^{*m-1} B_2^* f\| \, \|g\| + {}^mC_2 \|B_1^{*m-2} B_2^{*2} f\| \, \|g\| + $$
$$+ \cdots + \|B_2^{*m} f\| \, \|g\|$$

$$= \| B_1^m f \| \, \| g \| + {}^mC_1 \| B_1^{m-1} B_2 f \| \, \| g \| + \cdots + {}^mC_m \| B_2^m f \| \, \| g \|$$

$$\leqslant \frac{\varepsilon^m}{2^m} \| f \| \, \| g \| + {}^mC_1 \frac{\varepsilon^{m-1}}{2^{m-1}} \| B_2 f \| \, \| g \| + {}^mC_2 \frac{\varepsilon^{m-2}}{2^{m-2}} \| B_2^2 f \| \, \| g \| +$$

$$+ \cdots + {}^mC_m \frac{\varepsilon^m}{2^m} \| f \| \, \| g \| \leqslant$$

$$\leqslant \frac{\varepsilon^m}{2^m} \| f \| \, \| g \| + {}^mC_1 \frac{\varepsilon^{m-1}}{2^{m-1}} \frac{\varepsilon}{2} \| f \| \, \| g \| + {}^mC_2 \frac{\varepsilon^{m-2}}{2^{m-2}} \frac{\varepsilon^2}{2^2} \cdot$$

$$\| f \| \, \| g \| + \cdots + {}^mC_m \frac{\varepsilon^m}{2^m} \| f \| \, \| g \|$$

$$= \frac{\varepsilon^m}{2^m} \| f \| \, \| g \| + \frac{\varepsilon^m}{2^m} [{}^mC_1 + {}^mC_2 + \cdots + {}^mC_m] \| f \| \, \| g \|$$

$$= \frac{\varepsilon^m}{2^m} \| f \| \, \| g \| + \frac{\varepsilon^m}{2^m} (2^m - 1) \| f \| \, \| g \|$$

$$= \varepsilon^m \| f \| \, \| g \|, \qquad \text{for all } m.$$

Thus $\langle f, g \rangle = 0$ and hence $H \subset E^{\perp}$. Since E is a reducing subspace for B_1 and B_2, $E^{\perp} = K$ and so $E = \{0\}$. It follows that each B_i is invertible. Hence $0 = (0, 0) \notin \Pi \, \sigma(B_i) \supseteq \sigma(B)$.

Corollary 1. *If f is a rational function without singularities on $\sigma(A)$, then $\| f(A) \| = \| f(B) \|$.*

Proof. By Theorem 1, f is a rational function without singularities on $\sigma(B)$. Thus $f(B)$ is defined. Since $\sigma(B) \subset \sigma(A)$, it follows that $f(\sigma(B)) \subset \subset f(\sigma(A)$ i.e. $\sigma(f(B)) \subset \sigma(f(A))$. Now $f(B)$ is normal, and hence

$$\| f(B) \| = r(f(B)) \leqslant r(f(A)) \leqslant \| f(A) \|,$$

where $r(T)$ denotes the spectral radius of the operator T. As $\| f(A) \| \leqslant \| f(B) \|$, our result follows.

Corollary 2. *The joint spectrum $\sigma(A)$ is a joint spectral set for a commuting family $A = (A_1, \ldots, A_n)$ of subnormal operators on H.*

P r o o f. This follows from the fact that any closed super set of a joint spectral set is a joint spectral set and the Corollary 1.

Dash [5] has proved that if $A=(A_1, \ldots, A_n)$ is a commuting family of normal operators on H then $\overline{W(A)}$ is the closed convex hull of the joint spectrum $\sigma(A)$. We shall prove that if we take $A_1, \ldots, A_n$ to be subnormal operators on H_i's and define T_k's as before, then $\overline{W(T)}$ is the closed convex hull of $\sigma(T)$.

Let $B_1, \ldots, B_n$ be the minimal normal extensions of $A_1, \ldots, A_n$ on Hilbert spaces $K_1, K_2, \ldots, K_n$ respectively such that each H_i is a subspace of K_i. If we define

$$S_k = I_1'\otimes I_2'\otimes \cdots \otimes I_{k-1}'\otimes B_k\otimes I_{k+1}'\otimes \cdots \otimes I_n'$$

on the tensor product space $K_1\otimes K_2\otimes \cdots \otimes K_n$, where I_i' is the identity operator on K_i, $1\leqslant i\leqslant n$, then it can be easily seen that $S_1, S_2, \ldots, S_n$ are the minimal normal extensions of $T_1, \ldots, T_n$.

T h e o r e m 2. *If $A_1, \ldots, A_n$ are subnormal operators on $H_1, \ldots, H_n$ with the minimal normal extensions $B_1, \ldots, B_n$ on Hilbert spaces $K_1, \ldots, K_n$ respectively, then $\overline{W(T)}=$ closed conv $\sigma(T)$, where conv denotes the convex hull.*

P r o of. Using Theorem 1.

$$\sigma(S_i, \ldots, S_n)\subset\sigma(T_1, \ldots, T_n)$$

Also

$$W(T)=\Pi\, W(T_i)=\Pi\, W(A_i)\subset\Pi\, W(B_i)=W(S_1, \ldots, S_n)$$

Therefore,

$$\begin{aligned}\overline{W(S_1, \ldots, S_n)} &= \text{closed conv } \sigma(S_1, \ldots, S_n)\\ &\subset \text{closed conv } \sigma(T_1, \ldots, T_n)\\ &\subset \overline{W(T_1, \ldots, T_n)}\\ &\subset \overline{W(S_1, \ldots, S_n)}.\end{aligned}$$

C o r o l l a r y 3. *The closure of the joint numerical range of an n-tuple of commuting subnormal operators on the tensor product space is the same as the closure of the joint numerical range of their minimal normal extensions.*

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