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A NOTE ON ULTRAPOWER CARDINALITY

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Here we give some evaluations of ultrapower cardinality, obtained by generalising techniques developed in [1] and [2]. Filter regularity definition is from [2].

We say that a filter D is (α, β)-regular if D contains a family E such that

$$|E| = \beta \quad \text{and}$$

$$X \subseteq E \;\&\; |X| \geqslant \alpha \rightarrow \cap X = \varnothing .$$

Ultrafilters D and E are equally regular if for all α and β D is (α, β)-regular iff E is (α, β)-regular.

In the following text we assume that D is an uniform ultrafilter over an arbitrary cardinal γ.

P r o p o s i t i o n. If D is (α, β) *regular and* ν, k *are cardinals such that*

$$\alpha \leqslant \nu \leqslant \beta \quad \alpha \leqslant k \leqslant \beta \quad \textit{and}$$

$$k^{\underset{\smile}{\nu}} = k, \qquad \textit{then}$$

1. $$\Big|\prod_D k\Big| \geqslant 2^{\beta} \qquad \textit{and}$$

2. $$\Big|\prod_D k\Big|^{\nu} = \Big|\prod_D k\Big| .$$

C o r o l l a r y.

3. *If* $(cf\gamma)^{\underset{\smile}{cf\gamma}} = cf\gamma$ *then* $\Big|\prod_D cf\gamma\Big|^{cf\gamma} = \Big|\prod_D cf\gamma\Big|$.

4. *If* γ *is strongly inaccessible or* $(\gamma = \lambda^{+}$ *and* $2^{\lambda} = \lambda^{+})$ *then*

$$\Big|\prod_D \gamma\Big| = 2^{\gamma}.$$

Proof.

1. By assumption, $k^{\underset{\smile}{\nu}}=k$, what implies that $|\prod_D k|=|\prod_D k^{\underset{\smile}{\nu}}|$ and $\nu\leqslant k$. Since D is (α,β)-regular and $\nu\in[\alpha,\beta]$, D is (ν,β)-regular too. Let $E\subset D$ be such that

a) $|E|=\beta$

b) if $X\subset E$ and $|X|\geqslant\nu$ then $\cap X=\varnothing$.

Let E be well ordered by $\leqslant$. For all $i\in\cup E\in D$ we define:

$$X(i)=\{e\in E:\ i\in e\}\quad\text{and}$$

$\mathrm{seq}(i)$ to be the sequence of $e\in X(i)$ ordered by $\leqslant$. Given any $g\in{}^E\nu$, we define g'

if $\mathrm{seq}(i)=\langle e_\xi\mid\xi<\nu_i<\nu\rangle$ then

$$g'(i)=\langle g(e_\xi)\mid\xi<\nu_i<\nu\rangle.$$

$X(i)\subset E$ and $i\in\cap X(i)$, hence by b), $|X(i)|<\nu$ and thus $g'(i)\in k^{\underset{\smile}{\nu}}$. Define $\pi\colon{}^E\nu\to\prod_D k^{\underset{\smile}{\nu}}$ with $\pi g=g'_D$. We prove that π is 1—1. Let $g,h\in{}^E\nu$ and $g\neq h$. That means that for some $e\in E$ $g(e)\neq h(e)$. Let e be e_λ in $\mathrm{seq}(i)$. Hence, for all $i\in e$

$$g'(i)=\langle g(e_i)\cdots g(e_\lambda)\cdots g(e_\xi)\cdots\mid\xi<\nu_i<\nu\rangle\neq\langle h(e_i)\cdots h(e_\lambda)\cdots h(e_\xi)\cdots$$
$$\cdots\mid\xi<\nu_i<\nu\rangle=h'(i).$$

Since $e\in D$, we have that $g'_D\neq h'_D$. Hence

$$|\prod_D k|\geqslant|{}^E\nu|=2^\beta.$$

2. $k^{\underset{\smile}{\nu}}=k$ is assumed. So, $|\prod_D k|=|\prod_D k^{\underset{\smile}{\nu}}|$. We shall prove that $|\prod_D k|^\nu\leqslant|\prod_D k^{\underset{\smile}{\nu}}|$. To do that it is sufficient to find τ, which maps a subset of $\prod_D k^{\underset{\smile}{\nu}}$ onto ${}^\nu(\prod_D k)$. For this, it is sufficient to find a $\sigma\colon{}^\nu({}^\gamma k)\to{}^\gamma(k^{\underset{\smile}{\nu}})$ such that

(+) if $g,h\in{}^\nu({}^\gamma k)$ and $\sigma g=_D\sigma h$, then for all $\xi<\nu$, $g(\xi)=_D h(\xi)$.

Since then τ can be defined by:

if $\sigma g=f$ then $\tau(f_D)=\langle g(\xi)\mid\xi<\nu\rangle$.

D is (α,β)-regular and hence (α,ν)-regular, so there is an $E\subset D$ such that: $|E|=\nu$, and for all $X\subset E$, $|X|\geqslant\alpha$ implies that $\cap X=\varnothing$. Let E be well ordered with $\leqslant$. From here $E=\{e_\xi\mid\xi<\nu\}$. By choice of E, for any $i\in\cup E$ there is $\nu_i<\nu$ such that $i\in e_{\nu_i}$ but for all $\xi>\nu_i$, $i\notin e_\xi$. Now we define σ:

if $g\in{}^\nu({}^\gamma k)$ then $(\sigma g)(i)=\langle g(\varnothing)(i)\cdots g(\xi)(i)\cdots\mid\xi\leqslant\nu_i\rangle$.

$\sigma g\in{}^\gamma(k^{\underset{\smile}{\nu}})$. We prove (+). Let $\sigma g=_D\sigma h$ and let $X=\{i\in\gamma\colon(\sigma g)(i)=(\sigma h)(i)\}$. $X\in D$. For all $\xi<\nu$ we define $d_\xi=\{i\in\gamma\colon\nu_i>\xi\}$. Since $\{i\in\gamma\colon\nu_i>\xi\}=\bigcup_{\lambda>\xi}e_\lambda$, we see that for any $\xi<\nu$, $d_\xi \in D$, and $\gamma \setminus d_\xi \notin D$. Hence, for all $\xi<\nu$, $d_\xi \cap X \in D$. However, for all $i \in d_\xi \cap X$ we have that $\nu_i > \xi$ and $(\sigma g)(i) = (\sigma h)(i)$. Thus,

$$\{i \in \gamma : g(\xi)(i) = h(\xi)(i)\} \supset d_\xi \cap X \in D,$$

which implies that $g(\xi) =_D h(\xi)$.

3. D is a uniform ultrafilter over γ therefore it is $(cf(\gamma), cf(\gamma))$-regular. $cf(\gamma)^{\underline{cf(\gamma)}} = cf(\gamma)$ is assumed, thus we can apply 2. to obtain

$$\left|\prod_D cf(\gamma)\right|^{cf(\gamma)} = \left|\prod_D cf(\gamma)\right|.$$

4. a) if γ is strongly inaccessible then $cf\gamma = \gamma$ and $\gamma^{\underline{\gamma}} = \gamma$. Thus, we have

$$\left|\prod_D \gamma\right| = \left|\prod_D \gamma\right|^\gamma = 2^\gamma.$$

b) if γ is a successor cardinal, say $\gamma = \lambda^+$, then $cf\gamma = \gamma$. Suppose that continuum hypothesis is true on λ i.e. $2^\lambda = \lambda^+ = \gamma$. Then we have $\gamma^{\underline{|\gamma|}} = \gamma^\lambda = 2^\lambda = \gamma$, and, by 3.

$$\left|\prod_D \gamma\right| = \left|\prod_D \gamma\right|^\gamma = 2^\gamma.$$

From Frayne, Morell and Scott Theorem we know that $\left|\prod_D \gamma\right| > \gamma$. Now we see that in the cases a) and b) we have $\left|\prod_D \gamma\right| = 2^\gamma$ or, equally $\left|\prod_D \gamma\right|$ does not depend on the continuum hypothesis at γ.

We could not answer the following question.

Is it possible that for some cardinal k and some equally regular ultrafilters D and E

$$\left|\prod_D k\right| \neq \left|\prod_E k\right| \ ?$$

REFERENCES

[1] Chang, Keisler, Model theory, North Holland, 1973.
]2] Comfort, Negrepontis, The theory of ultrafilters, Springer Verlag 1974.