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THE EQUIVALENCE OF CERTAIN LINEAR AND NONLINEAR DIFFERENTIAL EQUATIONS OF NTH ORDER

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1.0. Introduction

Nonlinear differential equations (n.d.e.) usually have particular solutions (see, ex. [1—6]).

The theory of linear d.e. is rather advanced. It has been developing continuously, starting from classical works up to contemporary researches under the influence of analysis, geometry, physics and other sciences. The theory achieved great superposition success mostly because of the superposition principle, about which the following theorem speaks:

If $g_i (i=1, \ldots, n)$ is a solution of a certain homogenous l.d.e., then $\sum_{i=1}^{n} g_i C_i$, where C_i — are some arbitrary constants, is also a solution of this d.e.

This is incorrect in the case of n.d.e.. So naturally, there is a tendency to attempt to tranform n.d.e. to the linear and conversely [..., 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 21, 22].

In the present work almost all of the problems concerning the connections between the theory of linear and n.d.e. of any order are included, and some of the answer to the question of J. M. Thomas (see [91]) about the d.e. of n order is given. The central place in this work is the consideration of d.e. of nth order of a general form.

1.1. Differential Equations of n-th order. Let $g(x)$ be the general solution of the l.d.e.

$$(1.1) \qquad \sum_{r=0}^{n} a_r(x)\, g_{(x)}^{(r)} = h(x), \qquad (g_{(x)}^{(0)} = g(x),$$

where the coefficients $a_r(x)\,(r=0, 1, \ldots, n)$, $h(x)$ are functions, in general complex and continuous in some domain.

The $g^{(r)}(x)$ are the derivatives of r th order $(r=1, \ldots, n)$ and naturally, $a_n(x) \neq 0$. As far as we deal with n.d.e., we try to place minumum conditions of smoothness on the coefficient $a_r(x)$. Usually, they are proposed to be infinitely differentiable.

Consider the n.d.e.

$$\Phi(x, y, y^{(1)}, \ldots, y^{(n)})=0, \tag{1.2}$$

that has the solution

$$y=f(g(x)) \tag{1.3}$$

where f is some function of x, which is to be found.

Suppose that $f(g)$ has continuous derivatives up to the nth order where n is a natural number.

Let us indicate

$$g_{(x)}^{(r)}=\frac{d^r(g(x))}{dx^r}, \tag{1.4}$$

$$f^{(r)}=\left[\frac{d^r f(\bar{y})}{d\bar{y}^r}\right]_{\bar{y}=g(x)}, \quad (r=\overline{1, n}). \tag{1.5}$$

Then we have from (1.3) to the derivative of nth order

$$y^{(n)}=\sum_{r=1}^{n} F_{n,r}(g^{(1)}, \ldots, g^{(n-1)})\, f^{(r)}, \tag{1.6}$$

or in the form

$$y^{(n)}=f^{(1)} g^{(n)}+\sum_{r=2}^{n} F_{n,r}(g^{(1)}, \ldots, g^{(n-1)})\, f^{(r)}, \tag{1.6'}$$

where $f^{(r)}$ are given in (1.5) and $F_{n,r}(g)$ are the derivatives of the function g.

The formula (1.6) can be written as

$$\sum_{r=1}^{n} F_{n,r}(g^{(1)}, \ldots, g^{(n)})\, t^r=\sum_{r=1}^{n} F_{n,r}(tg^{(1)}, \ldots, tg^{(n)}), \tag{1.7}$$

and

$$F_{n,r}(tg^{(1)}, \ldots, tg^{(n)})=\sum \frac{n!}{s_1! \ldots s_n!}\left(\frac{tg^{(1)}}{1!}\right)^{s_1} \cdot \left(\frac{tg^{(2)}}{2!}\right)^{s_2} \cdots \left(\frac{tg^{(n)}}{n!}\right)^{s_n}, \tag{1.8}$$

where

$$\begin{cases} \sum_{k=1}^{n} s_k=r, \\ \sum_{k=1}^{n} k s_k=n, \end{cases} \tag{1.9}$$

and S_k are nonnegative integers.

Putting (1.1) as

$$g_{(x)}^{(n)}=\frac{1}{a_n(x)}\left((h(x)-a_0(x)\,g(x)-\sum_{r=1}^{n-1}a_r(x)\,g_{(x)}^{(r)}\right), \tag{1.1'}$$

into (1.6′), and using (1.8):

$$a_n y^{(n)}=f^{(1)}(h-a_0 g)-f^{(1)}\cdot\sum_{r=1}^{n-1}a_r g^{(r)}+a_n\sum_{r=2}^{n}F_{n,r}(g^{(1)},\ldots,g^{(n-1)})f^{(r)}. \tag{1.10}$$

According to (1.6′), we obtain;

$$g^{(r)}=\frac{y^{(r)}-\sum_{l=2}^{r}F_{r,l}(g^{(1)},\ldots,g^{(r-1)})f^{(l)}}{f^{(1)}}, \tag{1.11}$$

where if $r<l$, $F_{r,l}(g^{(1)},\ldots,g^{(r-1)})=0$.
So we have from (1.11) and (1.10)

$$\sum_{r=1}^{n}a_r y^{(r)}+f^{(1)}(a_0 g-h)-\sum_{r=2}^{n}a_r\sum_{l=2}^{n}F_{r,l}(g^{(1)},\ldots,g^{(r-1)})f^{(l)}=0, \tag{1.12}$$

where $2\leqslant l\leqslant r$.

Making the substitution $Df^{(r)}=f^{(r+1)}$, $D=p\dfrac{\partial}{\partial y}$,

$$f^{(r+1)}=\Big(\underbrace{\Big(p\frac{\partial}{\partial y}\Big)}_{\frac{\partial}{\partial x}}p\Big):$$

$$f^{(r+1)}=\left(\frac{\partial}{\partial x}\right)^{r}p(y(x)),\ (r=0,\ldots,n-1). \tag{1.13}$$

From (1.13), when $r=0$, we have

$$g=\int\frac{dy}{p(y)}. \tag{1.14}$$

And from (1.14) we have:

$$g^{(n-1)}=\sum_{r=1}^{n-1}\left[\sum_{q=1}^{r-1}\left(\frac{1}{p}\right)_q\sum_{\substack{\Sigma s_k=q,\\ \Sigma ks_k=r-1,}}\frac{(r-1)!}{s_1!\ldots s_{r-1}!}\left(\frac{p^{(1)}}{1!}\right)\cdots\right. \tag{1.15}$$

$$\left.\cdots\left(\frac{p^{(r-1)}}{(r-1)!}\right)^{s_{r-1}}\right]\sum_{\substack{\Sigma t_k=r,\\ \Sigma kt_k=n,}}\frac{(n-1)!}{t_1!\ldots t_{n-1}!}\left(\frac{y^{(1)}}{1!}\right)^{t_1}\cdots\left(\frac{y^{(n-1)}}{(n-1)!}\right)^{t_{n-1}}$$

where

$$\left(\frac{1}{p}\right)_q = \frac{(-1)^q q!}{p^{q+1}}, \tag{1.16}$$

Putting (1.13), (1.14), (1.15) into (1.12) and denoting

$$\begin{cases} b_1(y) = p(y) \int \frac{1}{p(y)} dy, \\ b_2(y) = -\frac{p^1(y)}{p(y)}, \\ b_3(y) = \frac{2 p^{(1)}_{(y)} - p(y) p^{(2)}_{(y)}}{p^2(y)}, \\ b_4(y) = -6 \left(\frac{p^{(1)}_{(y)}}{p(y)}\right)^3 + 6 \frac{p^{(1)}_{(y)} p^{(2)}_{(y)}}{p^2_{(y)}} - \frac{p^{(3)}_{(y)}}{p(y)}, \\ b_5(y) = 24\left(\frac{p^{(1)}_{(y)}}{p^{(y)}}\right)^4 - 36 \frac{p^{(1)}_{(y)} p^{(2)}_{(y)}}{p^3_{(y)}} + 6 \frac{p^{(2)2}_{(y)}}{p^2_{(y)}} + 8 \frac{p^{(1)}_{(y)} p^{(2)}_{(y)}}{p^2_{(y)}} - \frac{p^{4(y)}_{(y)}}{p^{(y)}}, \end{cases} \tag{1.17}$$

we can write (1.12) as

$$\sum_{r=1}^{n} a_r y^{(r)} + \sum_{r=2}^{n} a_r \sum_{l=2}^{r} G_{r,l}(y^{(1)}, \ldots, y^{(r-1)}) b_r(y) + a_0(x) b_1(y) h(x) = 0, \tag{1.18}$$

where $G_{r,1}$ are derivatives of the function of y, for $h(x) \neq 0$.

$$b_0 = -p, \tag{1.19}$$

and functions $b_i(y)$, $i = 1, \ldots, n$ satisfy she following expression

$$b_1' + b_1 b_2 = 1, \quad b_i = b_{i-1}' + b_{i-1} b_2, \quad i = 3, 4, \ldots, n. \tag{1.20}$$

So we come to the result:

Theorem 1.1. *D.e. (1.18), where the functions $b_i(y)$ ($i = 1, \ldots, n$), are related by the condition (1.17), or (1.20), have the solution $y = f(g(x))$, where g is the general solution of (1.1), and f is defined from*

$$f' - p(f) = 0. \tag{1.21}$$

The study of equations of Cauchy leads to (1.18), so using this method, we can obtain the solutions which satisfy the initial conditions. Concretely, if we have at first $g_0, g_0^{(1)}, \ldots, g_0^{(n-1)}$, for (1.1), then from (1.14) and its derivative, we can define the first $y_0, y_0^{(1)}, \ldots, y_0^{(n-1)}$ of equation (1.18). The converse is also true.

For some questions of the system of Cauchy, such as its unity, it is natural to use the modified theory of n.d.e.. Here we do not consider these problems.

Note. Now it is easy to check that because of (1.18), when $n=1, 2, 3, 4, 5$ and any a_r, b_r, h we generate the equations, that have been studied during the past 25 years and earlier (see for example, the d.e. like the ones in [6, 9, 20, 21, 22])

Several examples will now be considered. Here a, b, C_1 $(i=0, 1, \dots)$ are some constants.

Example 1.1. If $n=1$ then (1.18) becomes (as in [9])

$$a_1(x)\,y' + a_0(x)\,b_1(y) = b_0(y)\,h(x) = 0, \tag{1.22}$$

where

$$b_1(y) = b_0(y)\int \frac{1}{b_0(y)}\,dy, \quad b_0\left(\frac{b_1}{b_0}\right)' = 1. \tag{1.23}$$

Suppose in (1.22) $a_1 = 1$, $b_0(y) = y^n$, $n \neq -1$. Then from (1.23): $b_1(y) = \dfrac{1}{1-n}\,y$,

and (1.22) has the Bernoulli type: $y' + \dfrac{1}{1-n}\,a_0(x)\,y + h(x)\,y^{(n)} = 0$.

Taking into consideration that different forms of a_r, b_r, $r=0, 1, 2)$, $h(x)$ and $n=1$ in (1.18), (1.20) and so on in (1.1) we can have any d.e. of the first order. Then in case of some special choice of a_r, b_r, h it is possible to have known equations. For example, see [19, pp. 294—362].

Example 1.2. Using (1.18), (1.20) when $n=2$, $h=0$, $a_2=1$, $b_2(y)=\dfrac{a}{y}$, equation (1.18) looks like Painlevé's [20],

$$yy'' + a_1(x)\,yy' + ay^{(1)\,2} + \frac{a_0(x)\,y^2}{1+a} = 0, \; a = \mathrm{const} \neq -1. \tag{1.24}$$

So in some special cases of a_r, b_r, h, $n=2$ in (1.18), (1.20), and (1.1), (1.18) are connected with (1.20) leading to known equations (see [18, pp. 485—524]).

Example 1.3. With $n=3$ equation (1.18) becomes

$$a_3\,y^{III} + a_2\,y^{II} + a_1\,y^{I} + a_2\,G_{2,2}\,b_2 + a_3\,(G_{3,2}\,b_2 + G_{3,3}\,b_3) + a_0\,b_1 + b_0\,h = 0. \tag{1.25}$$

If, for example, $a_3=1$, $h=0$, then (1.25) becomes [21, eq. (6)]

$$y^{III} + a_2\,y^{II} + a_1\,y^{I} + b_3\,y^{(1)\,3} + a_2\,b_2\,y^{(1)\,2} + 3\,b_2\,y^{I}\,y^{II} + a_0\,b_1 = 0. \tag{1.26}$$

But for some $a_r(x)$, $b_r(y)$ $(r=0, 1, 2, 3)$, $h(x)$ in (1.18), (1.20), we can choose coefficients so that (1.18) would be a special case of the example in [19, see pp. 525—529].

Example 1.4. a) For $n=4$ in (1.18) we have [22, eq. (2.18)]:

$$\begin{aligned} a_4\,y^{IV} + a_3\,y^{III} + a_2\,y^{II} + a_1\,y^{I} + a_2\,G_{2,2}\,b_2 + a_3\,(G_{3,2}\,b_2 + G_{3,3}\,b_3) + \\ + a_4\,(G_{4,2}\,b_2 + G_{4,3}\,b_3 + G_{4,4}\,b_4) + a_0\,b_1 + b_0\,h = 0, \end{aligned} \tag{1.27}$$

that is

$$(1.28)\qquad a_4 y^{IV}+a_3 y^{III}+a_2 y^{II}+a_1 y^{I}+a_2 b_2 y^{(1)2}+a_3(3 b_2 y^{I} y^{II}+b_3 y^{(1)3})+$$
$$+a_4(4 b_2 y^{I} y^{III}+3 b_2 y^{(2)2}+6 b_3 y^{(1)} y^{II}+b_4 y^{(1)4})+a_0 b_1+b_0 h=0.$$

When $b_2=a/by$, $b\neq 0$, $a_i=0$ $(i=0, 1, 2, 3)$, $a_4=1$, $h=0$ in (1.28)

$$(1.29)\qquad b^3 y^3 y^{IV}+(2ab^2-3a^2 b+a^3)\, y^{(1)4}+6ab(a-b)\, yy^{(1)2} y^{II}+$$
$$+ab^2 y^2(3 y^{(2)2}+4 y^{I} y^{III})=0,$$

with the solution according to (1.21), $f(g)=\left(\dfrac{a+b}{b}g\right)^{b/(a+b)}$. Recalling that the solution of (1.1), in this case $y^{IV}=0$, $g=C_1 x^3+C_2 x^2+C_3 x+C_4$, we have a general solution:

$$y=(C_1 x^3+C_2 x^2+C_3 x+C_4)^{b/(a+b)}$$

When $a=b$ from (1.29):

$$(1.30)\qquad yy^{IV}+3 y^{(2)2}+4 y^{I} y^{III}=0,$$

with the solution $y=(2(C+g))^{1/2}$.

b) Let $a_4=1$, a_3ax, $a_2=6a^2$, $a_1=4a^3x^3$, $a_0=a^4x^4$, $h=0$, $b_2=-\operatorname{tg} y$. Then the general solution of (1.1) is

$$g=\sum_{s=1}^{4} C_s e^{-\frac{1}{2}ax^2+r_s x}$$

where r_s is some root of $r^4-6ar^2+3a^2=0$ (if it is complex, we must use trigonometric functions). We have

$$y^{IV}+axy^{III}+6a^2 y^{II}+4a^3x^3 y^{I}+(\operatorname{tg} y)\, y^{(1)4}-(axy^{(1)3}+6 y^{(1)2} y^{II})-$$
$$-\operatorname{tg} y\cdot(6a^2 y^{(1)2}+3 y^{(2)2}+4 y^{I} y^{III})+a^4x^4\left(\operatorname{tg} y+\frac{k}{\cos y}\right)=0,$$

which has the solution $y=\arcsin(Cg+C_1)$. In case $a=0$, $b_2=1/y$, we have (1.30).

c) Let $a_4=1$, $a_3=a_2=a_1=a_0=0$, $h(x)=\cos^2 x$, then the solution of (1.1) is $g=x^4/48+x^2/8+(\cos 2x)/32$ which satisfies the initial conditions $g(0)=1/32$, $g^{I}(0)=0$, $g^{II}(0)=1/8$, $g^{III}(0)=0$. In this case (1.28), for $p(y)=1/y$, becomes

$$yy^{IV}+3 y^{(2)2}+4 y^{I} y^{III}-\cos 2x=0.$$

The solution $y=(2(g+C))^{1/2}$, and $C=0$, satisfies the initial conditions $y=4$, $y^{I}=0$, $y^{II}=1/32$. $y^{III}=0$.

And here for different a_r, b_r, h we can have different types of examples [19, pp.525—529].

Example 1.5. With $n=5$, $a_5=1$, $a_1=-1$, $a_4=a_3=a_2=0$, $h=0$, then one solution of l.d.e. (1.1) is $g_0=e^x+\cos x-2$ which satisfies the initial conditions $g(0)=0$, $g^I(0)=1$, $g^{II}(0)=0$, $g^{III}(0)=1$, $g^{IV}(0)=2$. Equation (1.18), for ex. for $p(y)=e^{-y}$ becomes

$$y^V-y^I+y^{(1)5}+10\,y^{(1)3}y^{II}+10\,y^{(1)2}y^{III}+15\,y^I y^{(2)2}+$$

$$+10\,y^{II}y^{III}+5\,y^I y^{IV}=0,$$

with the solution $y=\ln(g_0+C)$, for $C=1$. Using (1.6), (1.13) satisfies $y=0$, $y^I=1$, $y^{II}=-1$, $y^{III}=3$, $y^{IV}=-8$ as the initial conditions.

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