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INDUCED AND INTRINSIC CURVATURE TENSORS OF A SUBSPACE IN THE FINSLER SPACE

Irena Čomić

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Abstract. There are known formulas which give a connection between the induced and intrinsic curvature tensors of the subspace and curvature tensors of the surrounding space [1]. Here these relations are given involving only the scalar quantity $\overset{\nu}{N}$

$$\overset{\nu}{N}=\overset{\nu}{N}_i\left(B^i_{\alpha\beta}+\Gamma^{*i}_{jk}\,B^{jk}_{\alpha\beta}\right)l^\alpha\,l^\beta$$

and the induced and intrinsic connection coefficients, not the curvature tensors of the surrounding space.

The subspace $F_m(u, \dot u)$ of the Finsler space $F_n(x, \dot x)$ is given by the equations:

$$x^i=x^i(u^1, \quad u^2, \ \dots \ u^m) \qquad i=1, 2, \ \dots \ n, \qquad rank\left(\frac{\partial x}{\partial u}\right)=m.$$

Let us denote by B^i_α the tangent vectors and by $\underset{\nu}{N}^i$ the normal vectors of the subspace F_m, where

(1)

$$\text{(a)}\ B^i_\alpha=\frac{\partial x^i}{\partial u^\alpha}, \qquad \alpha, \beta, \gamma, \delta, \varepsilon, \iota, \varkappa=1, 2, \ \dots \ m,$$

$$\text{(b)}\ g_{ij}\left(x(u), B_\alpha\dot u^\alpha\right)\underset{\nu}{N}^i B^j_\alpha=0, \qquad \lambda, \mu, \nu, \sigma, \omega, \tau, \rho=m+1, \ \dots \ n,$$

$$\text{(c)}\ g_{ij}\left(x(u), B_\alpha\dot u^\alpha\right)\underset{\nu}{N}^i\underset{\mu}{N}^j=\delta_{\nu\mu}, \qquad i, j, k, l, m, n=1, 2, \ \dots \ n.$$

The components of DB^i_α and $D\underset{\nu}{N^i}$ with respect to the induced connection coefficients are given by:

$$D B^i_\alpha = (\overline{\Gamma}^{*\delta}_{\alpha\beta}\, du^\beta + A^\delta_{\alpha\beta}\, \overline{D}\, l^\beta)\, B^i_\delta + (\overline{\theta}^{*\mu}_{\alpha\beta}\, du^\beta + A^\mu_{\alpha\beta}\, \overline{D}\, l^\beta)\, \underset{\mu}{N^i}.$$

$$D \underset{\mu}{N^i} = (-\overline{\theta}^{*\delta}_{\mu\beta}\, du^\beta - A^\delta_{\mu\beta}\, \overline{D}\, l^\beta)\, B^i_\delta + (\overline{\lambda}^{*\nu}_{\mu\beta}\, du^\beta + \overline{A}^\nu_{\mu\beta}\, \overline{D}\, l^\beta)\, \underset{\nu}{N^i}$$

and with respect to the intrinsic connection coefficients are given by:

$$D B^i_\alpha = [(\Gamma^{*\delta}_{\alpha\beta} + \Lambda^\delta_{\alpha\beta})\, du^\beta + A^\delta_{\alpha\beta}\, D\, l^\beta]\, B^i_\delta + (\theta^{*\mu}_{\alpha\beta}\, du^\beta + A^\mu_{\alpha\beta}\, D\, l^\beta)\, \underset{\mu}{N^i},$$

$$D \underset{\mu}{N^i} = (-\theta^{*\delta}_{\mu\beta}\, du^\beta + A^\delta_{\mu\beta}\, D\, l^\beta)\, B^i_\delta + (\lambda^{*\nu}_{\mu\beta}\, du^\beta + A^\nu_{\mu\beta}\, D\, l^\beta)\, \underset{\nu}{N^i}.$$

Given a vector field $\xi^i(x, \dot{x})$ defined on the subspace F_m i. e. for which

$$\xi^i(x, \dot{x}) = \xi^i(x(u),\, B_\alpha \dot{u}^\alpha),$$

then we may write

$$\xi^i = \xi^{i'} + \xi^{i''} \qquad \xi^{i'} = B^i_\alpha\, \xi^\alpha, \qquad \xi^{i''} = \underset{\mu}{N^i}\, \xi^\mu.$$

Above, the line elements are omitted, but it should be understood to be the line element $(x(u),\, B_\alpha \dot{u}^\alpha)$.

Let us consider the line elements P, P_1, P_2, P_3, P_3' where

$$P = (u, \dot{u}), \quad P_1 = (u + du,\, \dot{u} + d\dot{u}), \quad P_2 = (u + \delta u,\, \dot{u} + \delta\dot{u})$$

$$P_3 = (u + du + \delta u + \delta\, du, \quad \dot{u} + d\dot{u} + \delta\dot{u} + \delta\, d\dot{u})$$

$$P_3' = (u + \delta u + du + d\,\delta u, \quad \dot{u} + \delta\dot{u} + d\dot{u} + d\,\delta\dot{u})$$

If we move $\xi^i(u, \dot{u})$ parallel along $P P_2 P_3' P_3$ and also along $P P_1 P_3$ we get at P_3 the difference vector denoted by $\mathbf{D}\,\xi^i$, where

$$\mathbf{D}\,\xi^i = \mathbf{D}(\xi^{i'} + \xi^{i''}) = \mathbf{D}\,\xi^{i'} + \mathbf{D}\,\xi^{i''}. \tag{2}$$

$\mathbf{D}\,\xi^{i'}$ and $\mathbf{D}\,\xi^{i''}$ can be expressed in terms of the induced and intrinsic curvature tensors of the subspace. In terms of the induced curvature tensors these expressions have the form:

$$
\begin{aligned}
\mathbf{D}\,\xi^{i'} = & \Big\{\Big(\frac{1}{2}\,\bar R^{\ \varepsilon}_{\alpha\beta\gamma} - \bar\theta^{*}{}_{\alpha}{}^{\mu}{}_{[\beta}\,\bar\theta^{*\varepsilon}{}_{|\mu|\gamma]}\Big)[du^\beta,\,\delta u^\gamma] + \\
& (\bar P^{\ \varepsilon}_{\alpha\beta\gamma} - \bar\theta^{*}{}^{\ \mu}_{\alpha\beta}\,A^{\ \varepsilon}_{\mu\gamma} + A^{\ \mu}_{\alpha\gamma}\,\bar\theta^{*\varepsilon}{}_{\mu\beta})\,[du^\beta,\,\bar\Delta\, l^\gamma] + \\
& \Big(\frac{1}{2}\,\bar S^{\ \varepsilon}_{\alpha\beta\gamma} - A^{\ \mu}_{\alpha\ [\beta}A^{\ \varepsilon}_{|\mu|\gamma]}\Big)\,[\bar D\, l^\beta,\,\bar\Delta\, l^\gamma]\Big\}\,\xi^\alpha B^i_\varepsilon + \\
& \Big\{\Big(\frac{1}{2}\,\bar R^{\ \mu}_{\alpha\beta\gamma} + \bar\theta^{*\nu}_{\alpha}{}_{[\beta}\bar\lambda^{*\mu}_{|\nu|\gamma]}\Big)\,[du^\beta,\,\delta u^\gamma] + \\
& (\bar P^{\ \mu}_{\alpha\beta\gamma} + \bar\theta^{*}{}^{\ \nu}_{\alpha\beta}\,\bar A^{\ \mu}_{\nu\gamma} - A^{\ \delta}_{\alpha\gamma}\,\bar\theta^{*}{}^{\ \mu}_{\delta\beta})\,[du^\beta,\,\bar\Delta\, l^\gamma] + \\
& \Big(\frac{1}{2}\,\bar S^{\ \mu}_{\alpha\beta\gamma} + A^{\ \nu}_{\alpha\ [\beta}\bar A^{\ \mu}_{|\nu|\gamma]}\Big)\,[\bar D\, l^\beta,\,\bar\Delta\, l^\gamma]\Big\}\,\xi^\alpha \underset{\mu}{N^i},
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
\mathbf{D}\,\xi^{i''} = & \Big\{\Big(-\frac{1}{2}\,\bar{\bar R}{}^{\alpha}{}_{\mu\beta\gamma} + \bar\theta^{*\alpha}{}_{\nu[\beta}\,\bar\lambda^{*\ \nu}_{|\mu|\gamma]}\Big)\,[du^\beta,\,\delta u^\gamma] + \\
& (\bar{\bar P}{}^{\alpha}{}_{\mu\beta\gamma} + \bar\theta^{*\alpha}{}_{\nu\beta}\,\bar A^{\ \nu}_{\mu\gamma} + \bar\theta^{*\delta}{}_{\mu\beta}\,A^{\ \alpha}_{\delta\gamma})\,[du^\beta,\,\bar\Delta\, l^\gamma] + \\
& \Big(-\frac{1}{2}\,\bar{\bar S}{}^{\alpha}{}_{\mu\beta\gamma} + A^{\alpha}{}_{\nu\ [\beta}A^{\ \nu}_{|\mu|\gamma]}\Big)\,[\bar D\, l^\beta,\,\bar\Delta\, l^\gamma]\Big\}\,\xi^\mu B^i_\alpha + \\
& \Big\{\Big(\frac{1}{2}\,\bar R^{\ \nu}_{\mu\beta\gamma} + \bar\theta^{*\delta}{}_{\mu\ [\gamma}\bar\theta^{*\ \nu}_{|\delta|\beta]}\Big)\,[du^\beta,\,\delta u^\gamma] + \\
& (\bar P^{\ \nu}_{\mu\beta\gamma} + A^{\delta}{}_{\mu\gamma}\,\bar\theta^{*\nu}_{\delta\beta} - \bar\theta^{*\delta}{}_{\mu\beta}\,A^{\ \nu}_{\delta\gamma})\,[du^\beta,\,\bar\Delta\, l^\gamma] + \\
& \Big(\frac{1}{2}\,\bar S^{\ \nu}_{\mu\beta\gamma} - A^{\delta}{}_{\mu\ [\beta}A^{\ \nu}_{|\delta|\gamma]}\Big)\,[\bar D\, l^\beta,\,\bar\Delta\, l^\gamma]\Big\}\,\xi^\mu \underset{\nu}{N^i}
\end{aligned}
\tag{4}
$$

All the quantities appearing in the above equations are defined in [2].* $\mathbf{D}\,\xi^{i'}$ and $\mathbf{D}\,\xi^{i''}$ in terms of the intrinsic curvature tensors have the form:

(*) In [2] the definition of the tensor $\bar{\bar P}_{\alpha\mu\beta\gamma}$ should be changed so that the second equation of (2.34) takes the form:

$$\bar P_{\alpha\mu\beta\gamma} = \bar{\bar P}_{\alpha\mu\beta\gamma}.$$

$$\mathbf{D}\,\xi^{i'}=\Bigg\{\Bigg(\frac{1}{2}\,R^{\varepsilon}_{\alpha\beta\gamma}+\frac{1}{2}\,\hat{K}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma}+\Lambda^{\delta}_{\alpha\,[\beta}\Gamma^{*}{}_{|\delta|}{}^{\varepsilon}{}_{\gamma]}-$$

$$-\Lambda_{\delta}{}^{\varepsilon}{}_{[\beta}\Gamma^{*}{}_{|\alpha|}{}^{\delta}{}_{\gamma]}-\theta^{*}_{\alpha}{}^{\mu}{}_{[\beta}\theta^{*}{}^{\varepsilon}{}_{|\mu|\gamma]}\Bigg)\,[du^{\beta},\,\delta u^{\gamma}]+$$

$$(P^{\varepsilon}_{\alpha\beta\gamma}+\Lambda^{\varepsilon}_{\alpha\beta}\,|_{\gamma}+\Lambda^{\varepsilon}_{\alpha\varkappa}A^{\varkappa}{}_{\gamma}-\theta^{*}{}^{\mu}_{\alpha\beta}A^{\varepsilon}_{\mu\gamma}+A^{\mu}_{\alpha\gamma}\theta^{*}{}^{\varepsilon}{}_{\mu\beta})\,[du^{\beta},\,\Delta l^{\gamma}]+$$

(5) $$\Bigg(\frac{1}{2}\,S^{\varepsilon}_{\alpha\beta\gamma}-A_{\alpha}{}^{\mu}{}_{[\beta}A_{|\mu|}{}^{\varepsilon}{}_{\gamma]}\Bigg)[D\,l^{\beta},\,\Delta l^{\gamma}]\Bigg\}\,B^{i}_{\varepsilon}\,\xi^{\alpha}+$$

$$\Bigg\{\Bigg(\frac{1}{2}\,R^{\mu}_{\alpha\beta\gamma}+\Lambda^{\delta}_{\alpha\,[\beta}\,\theta^{*}{}^{\mu}_{|\delta|\gamma]}+\theta^{*}_{\alpha}{}^{\nu}[_{\beta}\,\lambda^{*}{}_{\nu}{}^{\mu}{}_{\gamma]}\Bigg)\,[du^{\beta},\,\delta u^{\gamma}]+$$

$$(P^{\mu}_{\alpha\beta\gamma}+\theta^{*\nu}_{\alpha\beta}A^{\mu}_{\nu\gamma}-A^{\delta}_{\alpha\gamma}\theta^{*\mu}_{\delta\beta}+\Lambda^{\delta}_{\alpha\beta}A^{\mu}_{\delta\gamma})\,[du^{\beta},\,\Delta l^{\gamma}]+$$

$$\Bigg(\frac{1}{2}\,S^{\mu}_{\alpha\beta\gamma}+A_{\alpha}{}^{\nu}{}_{[\beta}A_{|\nu|}{}^{\mu}{}_{\gamma]}\Bigg)\,[D\,l^{\beta},\,\Delta l^{\gamma}]\Bigg\}\,\underset{\mu}{N}{}^{i}\,\xi^{\alpha}$$

$$\mathbf{D}\,\xi^{i''}=\Bigg\{\Bigg(-\frac{1}{2}\,\underline{R}^{\alpha}{}_{\mu\beta\gamma}+\theta^{*}{}^{\varkappa}{}_{\mu[\beta}\Lambda^{\alpha}_{|\varkappa|\gamma]}+\theta^{*}{}^{\alpha}{}_{\nu[\gamma}\lambda^{*}{}^{\nu}_{|\mu|\beta]}\Bigg)\,[du^{\beta},\,\delta u^{\gamma}]+$$

$$(-\underline{P}^{\alpha}{}_{\mu\beta\gamma}+A^{\varkappa}{}_{\mu\gamma}\Lambda^{\alpha}_{\varkappa\beta}+\theta^{*}{}^{\varkappa}_{\mu\beta}A^{\alpha}_{\varkappa\gamma}+\theta^{*}{}^{\alpha}{}_{\nu\beta}A_{\mu}{}^{\nu}{}_{\gamma})\,[du^{\beta},\,\Delta l^{\gamma}]$$

$$\Bigg(-\frac{1}{2}\,\underline{S}^{\alpha}{}_{\mu\beta\gamma}+A^{\mu}_{\nu}{}_{[\beta}A^{\alpha}_{|\nu|\gamma]}\Bigg)\,[D\,l^{\beta},\,\Delta l^{\gamma}]\Bigg\}\,\xi^{\mu}\,B^{i}_{\alpha}+$$

(6) $$\Bigg\{\Bigg(\frac{1}{2}\,R_{\mu}{}^{\nu}{}_{\beta\gamma}-\theta^{*}{}^{\delta}{}_{\mu\,[\beta}\,\theta^{*}{}^{\nu}_{|\delta|\gamma]}\Bigg)\,[du^{\beta},\,\delta u^{\gamma}]+$$

$$(P_{\mu}{}^{\nu}{}_{\beta\gamma}+A^{\delta}{}_{\mu\gamma}\theta^{*}{}_{\delta}{}^{\nu}{}_{\beta}-\theta^{*}{}^{\delta}{}_{\mu\beta}A_{\delta}{}^{\nu}{}_{\gamma})\,[du^{\beta}\,\Delta l^{\gamma}]+$$

$$\Bigg(\frac{1}{2}\,S_{\mu}{}^{\nu}{}_{\beta\gamma}-A^{\delta}{}_{\mu\,[\beta}A_{|\delta|}{}^{\nu}{}_{\gamma]}\Bigg)[D\,l^{\beta},\,\Delta l^{\gamma}]\Bigg\}\,\xi^{\mu}\,\underset{\nu}{N}$$

All the quantities appearing in the former equations are defined in [3] (**) except $\hat{K}^{\varepsilon}_{\alpha\beta\gamma}$, where

(7) $$\frac{1}{2}\,\hat{K}^{\varepsilon}_{\alpha\beta\gamma}=\partial_{[\gamma}\Lambda_{|\alpha|}{}^{\varepsilon}{}_{\beta]}-\dot{\partial}_{\varkappa}\Lambda_{\varkappa}{}^{\varepsilon}{}_{[\beta}\Gamma^{*}_{\gamma}{}^{\varkappa}{}_{]}+\Lambda_{\alpha}{}^{\varkappa}{}_{[\beta}\Lambda_{|\varkappa|}{}^{\varepsilon}{}_{\gamma]}\,.$$

(**) The definition of the tensor $P_{\alpha\mu\beta\gamma}$ should be changed so that the second equation of (3.15) takes the form

$$P_{\alpha\mu\beta\gamma}=P_{\alpha\mu\beta\gamma}.$$

Corollary 1. The relation between $R^{\varepsilon}_{\alpha\beta\gamma}$ and $\bar{R}^{\varepsilon}_{\alpha\beta\gamma}$ in terms of the tensors $\bar{P}^{\varepsilon}_{\alpha\beta\gamma}$, $\bar{S}^{\varepsilon}_{\alpha\beta\gamma}$, $\hat{K}^{\varepsilon}_{\alpha\beta\gamma}$ and the intrinsic and induced connection coefficients of the subspace is given by:

$$\frac{1}{2}R^{\varepsilon}_{\alpha\beta\gamma}=\frac{1}{2}\bar{R}^{\varepsilon}_{\alpha\beta\gamma}-\bar{P}^{\varepsilon}_{\alpha\beta\delta}A^{\delta}{}_{\nu\gamma}\overset{\nu}{N}+$$

(8)
$$\frac{1}{2}\bar{S}^{\varepsilon}_{\alpha\delta\varkappa}A^{\delta}{}_{\nu\,[\beta}\overset{\nu}{N}A^{\varkappa}{}_{|\omega|\gamma]}\overset{\omega}{N}-\frac{1}{2}\hat{K}^{\varepsilon}_{\alpha\beta\gamma}$$

$$-\Lambda_{\alpha}{}^{\delta}{}_{[\beta}\Gamma^{*\varepsilon}_{|\delta|\gamma]}+\Lambda_{\delta}{}^{\varepsilon}{}_{[\beta}\Gamma^{*\delta}_{|\alpha|\gamma]}+\bar{\theta}^{*\mu}_{\alpha\gamma}A^{\varepsilon}{}_{\mu\delta}A^{\delta}_{\omega\beta}\overset{\omega}{N}-$$

$$-\bar{\theta}^{*\varepsilon}{}_{\mu\gamma}A^{\mu}_{\alpha\varkappa}A^{\varkappa}{}_{\nu\beta}\overset{\nu}{N}.$$

Proof. Using the relation

$$\bar{D}l^{\beta}=Dl^{\beta}-A^{\beta}{}_{\mu\gamma}\overset{\mu}{N}du^{\gamma}$$

and equating the coefficients of $[du^{\beta},\delta u^{\gamma}]B^{i}_{\varepsilon}\xi^{\alpha}$ in (3) and (5) we get:

$$\frac{1}{2}R^{\varepsilon}_{\alpha\beta\gamma}+\frac{1}{2}\hat{K}^{\varepsilon}_{\alpha\beta\gamma}+\Lambda_{\alpha}{}^{\delta}{}_{[\beta}\Gamma^{*\varepsilon}_{|\delta|\gamma]}-\Lambda_{\delta}{}^{\varepsilon}{}_{[\beta}\Gamma^{*\delta}_{|\alpha|\gamma]}-\theta^{*}_{\alpha}{}^{\mu}{}_{[\beta}\theta^{*\varepsilon}{}_{,|\mu|\gamma]}=$$

(9)
$$\frac{1}{2}\bar{R}^{\varepsilon}_{\alpha\beta\gamma}-\bar{\theta}^{*}_{\alpha}{}^{\mu}{}_{[\beta}\bar{\theta}^{*\varepsilon}{}_{|\mu|\gamma]}+\left(\bar{P}^{\varepsilon}_{\alpha\beta\delta}-\bar{\theta}^{*}{}^{\mu}_{\alpha\beta}A^{\varepsilon}{}_{\mu\delta}+A^{\mu}_{\alpha\delta}\bar{\theta}^{*\varepsilon}{}_{\mu\beta}\right)\left(-A^{\delta}_{\nu\gamma}\overset{\nu}{N}\right)+$$

$$\left(\frac{1}{2}\bar{S}^{\varepsilon}_{\alpha\delta\varkappa}-A^{\mu}_{\alpha\delta}A^{\varepsilon}{}_{\mu\varkappa}\right)\left(-A^{\delta}{}_{\nu\,[\beta}\overset{\nu}{N}\right)\left(-A^{\varkappa}{}_{|\omega|\gamma]}\overset{\omega}{N}\right)$$

If we use the formula

(10)
$$\theta^{*}{}^{\mu}_{\alpha\beta}=\bar{\theta}^{*}{}^{\mu}_{\alpha\beta}-A^{\mu}_{\alpha\varkappa}A^{\varkappa}{}_{\nu\beta}\overset{\nu}{N}$$

the last term in 9 becomes:

(11)
$$A^{\mu}_{\alpha\delta}A^{\delta}{}_{\nu\,[\beta}\overset{\nu}{N}\left(-A^{\varepsilon}{}_{|\mu\varkappa}A^{\varkappa}{}_{\omega|\gamma]}\overset{\omega}{N}\right)=$$

$$=A^{\mu}_{\alpha\delta}A^{\delta}{}_{\nu\,[\beta}\overset{\nu}{N}\left(\theta^{*\varepsilon}{}_{|\mu|\gamma]}-\bar{\theta}^{*\varepsilon}{}_{|\mu|\gamma]}\right)$$

If we substitute (11) into (9), after some calculations applying (10) again we obtain (8).

Corollary 2. The relation between $P^{\varepsilon}_{\alpha\beta\gamma}$ and $\overline{P}^{\varepsilon}_{\alpha\beta\gamma}$ in the terms of the tensor $\overline{S}_{\alpha}{}^{\varepsilon}{}_{\gamma\varkappa}$ and intrinsic connection coefficients is given by:

$$P^{\varepsilon}_{\alpha\beta\gamma}=\overline{P}^{\varepsilon}_{\alpha\beta\gamma}+\overline{S}^{\varepsilon}_{\alpha\gamma\varkappa}A^{\varkappa}_{\nu\beta}\overset{\nu}{N}- \tag{12}$$
$$\Lambda^{\varepsilon}_{\alpha\delta}\big|_{\gamma}-\Lambda^{\varepsilon}_{\alpha\varkappa}A^{\varkappa}_{\beta\gamma}.$$

Proof. Equating the coefficients of $[du^{\beta},\Delta l^{\gamma}]B^{i}_{\varepsilon}\xi^{\alpha}$ in (3) and (5) we get:

$$P^{\varepsilon}_{\alpha\beta\gamma}+\Lambda^{\varepsilon}_{\alpha\beta}\big|_{\gamma}+\Lambda^{\varepsilon}_{\alpha\varkappa}A^{\varkappa}_{\beta\gamma}-\theta^{*}{}^{\mu}_{\alpha\beta}A^{\varepsilon}_{\mu\gamma}+A^{\mu}_{\alpha\gamma}\theta^{*}{}^{\varepsilon}_{\mu\beta}=$$
$$\overline{P}^{\varepsilon}_{\alpha\beta\gamma}-\bar{\theta}^{*}{}^{\mu}_{\zeta}A^{\varepsilon}_{\mu\gamma}+A^{\mu}_{\alpha\gamma}\bar{\theta}^{*}{}^{\varepsilon}_{\mu\beta}+ \tag{13}$$
$$(\overline{S}^{\varepsilon}_{\alpha\gamma\varkappa}-A^{\mu}_{\alpha\gamma}A^{\varepsilon}_{\mu\varkappa}+A^{\mu}_{\alpha\varkappa}A^{\varepsilon}_{\mu\gamma})\,(A^{\varkappa}_{\nu\beta}\overset{\nu}{N})$$

Using (10) again the last two terms in the above equation become

$$-A^{\mu}_{\alpha\gamma}A^{\varepsilon}_{\mu\varkappa}A^{\varkappa}_{\nu\beta}\overset{\nu}{N}+A^{\mu}_{\alpha\varkappa}A^{\varkappa}_{\nu\beta}\overset{\nu}{N}A^{\varepsilon}_{\mu\gamma}=$$
$$=A^{\mu}_{\alpha\gamma}(\theta^{*}{}^{\varepsilon}_{\mu\beta}-\bar{\theta}^{*}{}^{\varepsilon}_{\mu\beta})+(\bar{\theta}^{*}{}^{\mu}_{\alpha\beta}-\theta^{*}{}^{\mu}_{\alpha\beta})A^{\varepsilon}_{\mu\gamma}$$

Substituting the above expression into (13), we obtain (12).
The relation

$$S^{\varepsilon}_{\alpha\beta\gamma}=\overline{S}^{\varepsilon}_{\alpha\beta\gamma} \tag{14}$$

is obtained form (3) and (5) in the similar way.

Corollary 3. The relation between $R^{\mu}_{\alpha\beta\gamma}$ and $\overline{R}^{\mu}_{\alpha\beta\gamma}$ in terms of the tensors $\overline{P}^{\mu}_{\alpha\beta\delta}$, $\overline{S}^{\mu}_{\alpha\delta\varkappa}$ and the induced and intrinsic connection coefficients is given by:

$$\frac{1}{2}R^{\mu}_{\alpha\beta\gamma}=\frac{1}{2}\overline{R}^{\mu}_{\alpha\beta\gamma}-\overline{P}^{\mu}_{\alpha\beta\delta}A^{\delta}_{\nu\gamma}\overset{\nu}{N}+\frac{1}{2}\overline{S}^{\mu}_{\alpha\delta\varkappa}A^{\delta}_{\nu[\beta}A^{\varkappa}_{|\omega|\gamma]}\overset{\nu}{N}\overset{\omega}{N}-$$
$$\Lambda^{\delta}_{\alpha[\beta}\theta^{*\mu}_{|\delta|\gamma]}-A^{\nu}_{\alpha\varkappa}A^{\varkappa}_{\omega[\beta}\overset{\omega}{N}\overline{\lambda}^{*\mu}_{|\nu|\gamma]}- \tag{15}$$
$$\bar{\theta}^{*\nu}_{\alpha\gamma}\overline{A}^{\mu}_{\nu\delta}A^{\delta}{}_{\omega\beta}\overset{\omega}{N}+A^{\varkappa}_{\alpha\delta}A^{\mu}_{\varkappa\beta}A^{\delta}_{\nu\gamma}\overset{\nu}{N}$$

Proof. Equating the coefficients of $[du^{\beta}\delta u^{\gamma}]\xi^{\alpha}\underset{\mu}{N^{i}}$ in (3) and (5) we obtain:

$$
\begin{aligned}
&\frac{1}{2} R^{\mu}_{\alpha\beta\gamma} + \Lambda^{\delta}_{\alpha[\beta}\, \theta^{*\mu}_{|\delta|\gamma]} + \theta^{*\nu}_{\alpha\ [\beta}\, \lambda^{*\mu}_{|\nu|\gamma]} = \\
&\frac{1}{2} \bar{R}^{\mu}_{\alpha\beta\gamma} + \bar{\theta}^{*\nu}_{\alpha\ [\beta}\, \bar{\lambda}^{*\mu}_{|\nu|\gamma]} + \\
&(\bar{P}^{\mu}_{\alpha\beta\delta} + \bar{\theta}^{*\nu}_{\alpha\beta}\, \bar{A}^{\mu}_{\nu\delta} - A^{\varkappa}_{\alpha\delta} A^{\mu}_{\varkappa\beta})\ (-A^{\delta}_{\nu\gamma} \underset{\nu}{N}) + \\
&\left(\frac{1}{2} \bar{S}^{\mu}_{\alpha\delta\varkappa} + A^{\nu}_{\alpha\delta} \bar{A}^{\mu}_{\nu\varkappa}\right) (-A^{\delta}_{\nu}{}_{[\beta} \overset{\nu}{N})\ (-A^{\varkappa}_{|\omega|\gamma]} \overset{\omega}{N})
\end{aligned} \tag{16}
$$

Multiplying the right-hang sides of (10) and (17), where

$$
\lambda^{*\ \mu}_{\ \nu\gamma} = \bar{\lambda}^{*\ \mu}_{\ \nu\gamma} - \bar{A}^{\mu}_{\nu\delta} A^{\delta}_{\ \omega\gamma} \overset{\omega}{N} \tag{17}
$$

and substituting into (16), we get (15).

C o r o l l a r y 4. The relation between $P^{\mu}_{\alpha\beta\gamma}$ and $\bar{P}^{\mu}_{\alpha\beta\gamma}$ in terms of the tensor $\bar{S}^{\mu}_{\alpha\delta\gamma}$ and the induced and intrinsic connection coefficients is given by:

$$
\begin{aligned}
P^{\mu}_{\alpha\beta\gamma} = \bar{P}^{\ \mu}_{\alpha\beta\gamma} - \bar{S}^{\ \mu}_{\alpha\delta\gamma} A^{\delta}_{\ \nu\beta} \overset{\nu}{N} - \Lambda^{\delta}_{\alpha\beta} A^{\mu}_{\delta\gamma} + \\
+ A^{\delta}_{\alpha\gamma} A^{\mu}_{\delta\varkappa} A^{\varkappa}_{\ \omega\beta} \overset{\omega}{N} + A^{\nu}_{\alpha\gamma} \bar{A}^{\mu}_{\nu\delta} A^{\delta}_{\ \omega\beta} \overset{\omega}{N}.
\end{aligned} \tag{18}
$$

P r o o f. Equating the coefficients of $[du^{\beta}, \Delta l^{\gamma}]\, \xi^{\alpha} \underset{\mu}{N^{i}}$ in (3) and (5), using (10) and $A^{\mu}_{\nu\gamma} = \bar{A}^{\mu}_{\nu\gamma}$, we obtain (18).

It is obvious that

$$
S^{\mu}_{\alpha\beta\gamma} = \bar{S}^{\mu}_{\alpha\beta\gamma}. \tag{19}
$$

Since

$$
\bar{R}_{\alpha\mu\beta\gamma} = \bar{\bar{R}}_{\alpha\mu\beta\gamma}, \quad \bar{P}_{\alpha\mu\beta\gamma} = \bar{\bar{P}}_{\alpha\mu\beta\gamma}, \quad \bar{S}_{\alpha\mu\beta\gamma} = \bar{\bar{S}}_{\alpha\mu\beta\gamma}
$$

$$
R_{\alpha\mu\beta\gamma} = \underline{R}_{\alpha\mu\beta\gamma}, \quad P_{\alpha\mu\beta\gamma} = \underline{P}_{\alpha\mu\beta\gamma}, \quad S_{\alpha\mu\beta\gamma} = \underline{S}_{\alpha\mu\beta\gamma},
$$

by equating coefficients of $[du^{\beta}, \delta u^{\gamma}]\, \xi^{\mu} B^{i}_{\alpha}$, $[du^{\beta} \Delta l^{\gamma}]\, \xi^{\mu} B^{i}_{\alpha}$ and $[D l^{\beta}, \Delta l^{\gamma}]\, \xi^{\mu} B^{i}_{\alpha}$ in (4) and (6), we will get (15), (18) and (19).

C o r o l l a r y 5. The relation between $R^{\nu}_{\mu\beta\gamma}$ and $\bar{R}^{\nu}_{\mu\beta\gamma}$ in terms of the tensors $\bar{P}^{\nu}_{\mu\beta\delta}$, $\bar{S}^{\nu}_{\mu\delta\varkappa}$ and the induced and intrinsic connection coefficients is given by

$$
\begin{aligned}
\frac{1}{2} R^{\ \nu}_{\mu\beta\gamma} = \frac{1}{2} \bar{R}^{\ \nu}_{\mu\beta\gamma} - \bar{P}^{\ \nu}_{\mu\beta\delta} A^{\delta}_{\ \omega\gamma} \overset{\omega}{N} + \frac{1}{2} \bar{S}^{\ \nu}_{\mu\delta\varkappa} A^{\delta}_{\sigma[\beta} \overset{\sigma}{N} A^{\varkappa}_{|\omega|\gamma]} \overset{\omega}{N} \\
+ \bar{\theta}^{*\delta}_{\ \mu\gamma} A^{\nu}_{\delta\varkappa} A^{\varkappa}_{\ \omega\beta} \overset{\omega}{N} - A^{\varkappa}_{\ \mu\delta} A^{\delta}_{\ \omega\beta} \overset{\omega}{N} \bar{\theta}^{*\ \nu}_{\varkappa\gamma}.
\end{aligned} \tag{20}
$$

Proof. Equating coefficients of $[du^{\beta}\,\delta u^{\gamma}]\,\xi^{\mu}\,\underset{\nu}{N^{i}}$ in (4) and (6), and using (10), we get (20).

Corollary 6. The relation between the tensors $P^{\nu}_{\mu\beta\gamma}$ and $\overline{P}^{\nu}_{\mu\beta\gamma}$ is given by:

$$P^{\nu}_{\mu\beta\gamma}=\overline{P}^{\nu}_{\mu\beta\gamma}+\overline{S}^{\nu}_{\mu\gamma\varkappa}\,A^{\varkappa}_{\nu\beta}\,\overset{\nu}{N} \tag{21}$$

Proof. Equating coefficients of $[du^{\beta},\,\Delta\, l^{\gamma}]\,\xi^{\mu}\,\underset{\nu}{N^{i}}$ in (4) and (6), and using (10), we obtain (21). It is obvious that

$$S^{\nu}_{\mu\beta\gamma}=\overline{S}^{\nu}_{\mu\beta\gamma}.$$

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Faculty of Technical sciences
University Novi Sad