PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 23 (37) 1978, pp. 53—65

$\mathcal{C}_p$-MOVABLE AT INFINITY SPACES, COMPACT ANR DIVISORS AND PROPERTY UVW^n

Zvonko Čerin

(Received November 24, 1977)

1. Introduction

In recent years several authors considered the problem of finding when will the one-point compactification $M \cup \infty$ of a locally compact ANR space M also be an ANR. The present paper treats three topics related to this problem.

We fiist define in § 3 classes of weakly $\mathcal{C}_p$-movable, $\mathcal{C}_p$-movable, and strongly $\mathcal{C}_p$-movable at infinity non-compact locally compact spaces, for an arbitrary class $\mathcal{C}_p$ of pairs of topological spaces. The rest of § 3 contains examples illustrating those concepts. The following § 4 presents elementary theorems concerning those classes that resemble some results from [6]. In § 5 we restrict our attention to ANR spaces. Under that assumption we prove that $\mathcal{C}_p$-movability at ∞ and strong $\mathcal{C}_p$-movability at ∞ are equivalent and that for every $\mathcal{C}_p$-movable at ∞ ANR space M the Freudenthal end-point compactification FM (and, therefore also, the one-point compactification $M \cup \infty$) is an ANR space.

The second topic, ANR divisors, is considered in § 6. The notion of an ANR divisor was introduced by Hyman in [15]. A compactum X is an ANR divisor provided the space Y/X obtained from any ANR space Y, containing X as a closed subset, by shrinking X to a point (or, equivalently, that $(Y—X) \cup \infty$) is an ANR space. We prove that every pointed fundamental absolute neighorhood retract is an ANR divisor and establish a number of theorems showing that ANR divisors have many properties in common with fundamental absolute neighborhood retracts. An example due to Dydak shows that the class of ANR divisors is much wider than the class of pointed FANR's.

Finally, in § 7 we utilize results of previous sections to shed new light on Aımentrout's property UVW^n [1]. It turned out that the class of UVW^n-compacta, i. e., compacta that have property UVW^n in every ANR space, can be regarded as the n-dimensional approximation of the class of pointed fundamental absolute neighborhood retracts. Theorem 2 in [1] shows that under some restrictions an n-dimensional UVW^n-compactum is an ANR divisor. Here we prove that UVW^n-compacta of fundamental dimension $<n$ are actually pointed fundamental absolute neighborhood retracts.

This paper includes portions of Chapters V and VII of the authors's dissertation written under Prof. R. D. Anderson at LSU in 1975.

2. Preliminaries

We assume the reader is familiar with shape theory [2] and infinite-dimensional topology [8]. The method of presentation and ideas for proofs of some of our results are included in [5], [6], and [7] so that at some places details will be omitted.

Let M and N denote non-compact locally compact spaces. A continuous function $f: M \to N$ is a *proper map* provided $f^{-1}(A)$ is compact for every compact subset A of N. Proper maps $f, g: M \to N$ are *homotopic at* ∞ if for every compact $A \subset N$ there is a compact $B \subset M$ such that restrictions $f|_{M-B}$ and $g|_{M-B}$ are homotopic in $N-A$. We shall say that N *proper homotopy dominates* (*homotopy dominates at* ∞) M if there are proper maps $f: M \to N$ and $g: N \to M$ with $g \circ f$ and id_M properly homotopic (homotopic at ∞).

Throughout the paper $n \geqslant 0$ is an integer, $\mathcal{C}$ is an arbitrary class of topological spaces, and $\mathcal{C}_p$ a class of pairs (X, X_0), where X is a topological space and X_0 is its closed subset. By $\mathcal{C}^n$ we denote a subclass of $\mathcal{C}$ consisting of all spaces in $\mathcal{C}$ with (covering) dimension $\leqslant n$. Let $\mathcal{P}$, $\mathcal{P}_p$, $\mathcal{B}_p^n$, and $\mathcal{S}^n$ denote the class of all finite simplicial complexes, the class of all finite simplicial pairs, the class $\{(B^1, S^0), (B^2, S^1), \ldots, (B^n, S^{n-1})\}$, and the class $\{S^0, S^1, \ldots, S^n\}$, respectively, where B^k is the k-dimensional unit solid ball and S^{k-1} is its boundary $(k-1)$-sphere.

3. $\mathcal{C}_p$-movability at infinity — examples

The notion of a $\mathcal{C}$-movable at ∞ non-compact locally compact space from [6] is in this section generalized introducing $\mathcal{C}_p$-movable at ∞ spaces. Since we consider homotopies defined on topological pairs (X, X_0), there are three different conditions that we can impose on restricitions of those homotopies onto X_0. Thus, we get three versions of that concept called weakly $\mathcal{C}_p$-movable, $\mathcal{C}_p$-movable, and strongly $\mathcal{C}_p$-movable at ∞ spaces. In the same way as movable compacta are closely related via the complement theorem [6, Theorem (4.2)] to $\mathcal{P}$-movable at ∞ ANR's we observe that FANR's are related to weakly $\mathcal{P}_p$-movable at ∞ ANR's and pointed FANR's are related to $\mathcal{P}_p$-movable at ∞ ANR's. In presenting examples of spaces illustrating our definition we improve Mardešić's theorem [19] saying that an LC^{n-1} compactum of dimension $\leqslant n$ is movable.

(3.1) Definition. A non-compact locally compact space M is said to be

(a) *weakly* $\mathcal{C}_p$*-movable at* ∞,

(b) $\mathcal{C}_p$*-movable at* ∞,

(c) *strongly* $\mathcal{C}_p$*-movable at* ∞, provided that for every compact set $A \subset M$, there is a compact set $B \supset A$ with the property that for every compact subset $C \supset A$ we can find a compact set D containing $B \cup C$ such that given a map $f: (X, X_0) \to (M-B, M-D)$ of a pair (X, X_0) in $\mathcal{C}_p$ into $(M-B, M-D)$ there is a homotopy $h_t: X \to M-A$, $0 \leqslant t \leqslant 1$, with $h_0 = f$, $h_1(X) \subset M-C$, and

(a) $h_1 | X_0 = f | X_0$,

(b) $h_t(X_0) \subset M-C$, $0 \leqslant t \leqslant 1$, and $h_1 | X_0 = f | X_0$,

(c) $h_t | X_0 = f | X_0$, $0 \leqslant t \leqslant 1$,

respectively.

(3.2) Examples. (a). For a class $\mathcal{C}$ let $\mathcal{C}_p^{\varnothing}$ denote the class of pairs $(X, \varnothing)$ where $X \in \mathcal{C}$. Then M is $\mathcal{C}$-movable at ∞ if and only if M is $\mathcal{C}_p^{\varnothing}$-movable at ∞.

(b). For a class $\mathcal{C}$ let $\mathcal{C}_p^h$ denote the class of pairs $(X \times [0, 1], X \times \{0\} \cup X \times \{1\})$ where $X \in \mathcal{C}$. Then a space M is $\mathcal{C}$-calm at ∞ [7] if and only if M is weakly $\mathcal{C}_p^h$-movable at ∞.

(c). Let $\sigma = \{X_i, f_i\}$ be a direct sequence of finite polyhedra and let Map (σ) be its infinite mapping cylinder obtained by glueing mapping cylinders of maps f_i together. Then Map (σ) is strongly $\mathcal{C}_p$-movable at ∞, for every class $\mathcal{C}_p$.

(d). Let X be a closed subset of a compact space Y and assume that X satisfies the isotopy compression axiom I-Comp (X, Y) [21]. Then $Y-X$ is strongly $\mathcal{C}_p$-movable at ∞, for every class $\mathcal{C}_p$.

(e). Let a countable discrete group G acts semi-freely on the Hilbert cube Q with a unique fixed point x_0. Then the quotient space $M = (Q - \{x_0\})/G$ is strongly $\mathcal{B}_p^n$-movable at ∞, for all $n \geqslant 0$.

(3.3) Theorem. (a) *A Z-set X in a compact ANR space N is an FANR if and only if $M = N - X$ is weakly $\mathcal{P}_p$-movable at ∞.*

(b) *A Z-set X in a compact ANR space N is a pointed FANR if and only if $M = N - X$ is $\mathcal{P}_p$-movable at ∞.*

Proof. (a) A standard proof (see [6, Theorem (4.2)]) can be given using the fact that FANR's coincide with strongly movable compacta [2].

(b) As in [6, Theorem (3.2)], without loss of generality, we can assume $N = Q$, the Hilbert cube. By [21] X is a pointed FANR if and only if X has I-regular open neighborhoods. This clearly suffices (see Example (3.2) (d)).

The following theorem improves Mardešić's result [19] stating that LC^{n-1} compacta are n-movable because Z-sets in compact ANR's whose complements are $\mathcal{P}^n$-movable at ∞ are n-movable [6] and $\mathcal{P}_p^n$-movable at ∞ spaces are $\mathcal{P}^n$-movable at ∞.

(3.4) Theorem. *The complement $M = Q - X$ of a locally $(n-1)$-connected Z-set X in the Hilbert cube Q is $\mathcal{P}_p^n$-movable at ∞.*

Proof. Let $A \subset M$ be an arbitrary compactum and let $U \subset Q - A$ be an open neighbourhood of X such that $\mathcal{U} = \{Q - Cl(U), Q - A\}$ is an open cover of Q. Pick a refinement $\mathcal{W}$ of $\mathcal{U}$ such that $\mathcal{W}$-close maps into Q are $\mathcal{U}$-homotopic. Since X is an LC^{n-1} compactum, the inclusion $i: X \hookrightarrow Q$ is a strong local connection in dimension $n-1$ (see [16] for definitions and notation). Hence, there is a refinement $\mathcal{V}$ of $\mathcal{W}$ for which the assertion $E(\mathcal{V}, \mathcal{W}, n)$ holds ([16, Lemma 1]). In other words, given an at most n-dimensional finite simplicial complex K, a subcomplex L of K, and maps $g: L \to X$ and $h: K \to Q$ such that $h|_L = i \circ g$ and h maps every simplex σ of K into some member of the collection $\{V \in \mathcal{V} \mid V \cap X \neq \varnothing\}$, then there is an extension $h': K \to X$ of g such that for every simplex σ of K we can find $W \in \mathcal{W}$ with $i \circ h'(\sigma) \cup h(\sigma) \subset W$. The choice of $\mathcal{W}$ assures that h and h' are homotopic in $Q-A$. The compactum $B = Q - \bigcup\{V \in \mathcal{V} \mid V \cap X \neq \varnothing\}$ is the one we are looking for. Indeed, if C is any compactum in M containing A, select a compactum D with respect to $B \cup C$ in the way analogous to how B was chosen with respect to A. Let $\lambda_t: Q \to Q$, $0 \leqslant t \leqslant 1$, be a deformation with $\lambda_t(Q) \subset Q - X$, $\lambda_t(Q-A) \subset Q-A$, $\lambda_t(Q-C) \subset Q-C$, and $\lambda_t|_A = \mathrm{id}$, for all $t > 0$.

Suppose $f: (K, L) \to (M-B, M-D)$ is a map of a pair (K, L) of finite polyhedra, $\dim K \leqslant n$, into $(M-B, M-D)$. A restriction $f|_L$ is homotopic in $M-C$ to a map $\hat{f}: L \to X$ by the choice of D. Hence, by the homotopy extension theorem, f is homotopic in $(Q-B, Q-C)$ to map $h: (K, L) \to (Q-B, X)$. Then h is homotopic in $(Q-A, M-C)$ to a map $h': K \to X$, where $h'|_L = \hat{f}$. Let $f_t: (K, L) \to (Q, A, Q-C)$ be a homotopy connecting f and $\lambda_1 \circ h'$. The composition $\lambda_{t(1-t)} \circ f_t$ is a homotopy in $(M-A, M-C)$ joining f and $\lambda_1 \circ h'$. With the help of the homotopy extension theorem it is easy to see that this implies $Q-X$ is $\mathcal{P}^n_p$-movable at ∞.

The last example uses the concept of a $\mathcal{C}$-trivial at ∞ space introduced in [4]. A non-compact locally compact space M is *$\mathcal{C}$-trivial at* ∞ if for every compact subset A of M there is a larger compact set B such that each map $f: X \to M-B$ of $X \in \mathcal{C}$ into a component of $M-B$ is null-homotopic in $M-A$.

(3.5) Example. If a connected, locally connected, locally compact space M has finitely many ends and is $\mathcal{S}^n$-trivial at ∞, then M is $\mathcal{B}^n_p$-movable at ∞.

Proof. Without loss of generality, we can assume M has only one end. Let A be a compact subset of M. Pick $B \supset A$ such that $M-B$ is connected and every map $S^k \to M-B$, $0 \leqslant k \leqslant n$, is null-homotopic in $M-A$. If $C \supset A$ is a compact set, select a compact $D \supset B \cup C$ so that $M-D$ is connected and any every map $S^k \to M-D$, $0 \leqslant k \leqslant n$, is null-homotopic in $M-(B \cup C)$.

Now, consider a map $f: (B^k, S^{k-1}) \to (M-B, M-D)$, $0 \leqslant k \leqslant n$. By assumption, there is a homotopy $h_t: S^{k-1} \to M-C$ with $h_0 = f|S^{k-1}$ and $h_1(S^{k-1}) = p =$ $=$ point $\in M-D$. Define a map g on the boundary of $B^k \times I$ into $M-B$ by

$$g(x, t) = \begin{cases} p & t = 1, \\ h_t(x), & x \in S^{k-1}, \quad t \in I, \text{ and} \\ f(x), & t = 0. \end{cases}$$

The choice of B implies that g has an extension $G: B^k \times I \to M-A$. On the space $B^k \times \{1\} \cup S^{k-1} \times [1, 2]$ define a partial homotopy into $M-C$ of the constant map into p on $B^k \times \{1\}$ by taking h_{2-t} on $S^{k-1} \times [1, 2]$. By [13, p. 120], that partial homotopy extends to a homotopy $H: B^k \times [1, 2] \to M-C$. The composition of homotopies G and H shows that M is $\mathcal{B}^n_p$-movable at ∞.

Assuming, in addition, that the space M in (3.5) is locally n-connected, then by Theorem (3.4) in [4] M is actually $\mathcal{P}^n$-trivial at ∞ and the above proof can be applied to get that M is $\mathcal{P}^n_p$-movable at ∞. Imposing even stronger condition that M is an ANR, by theorems (3.6) and (5.1) below, we can directly conclude that M is strongly $\mathcal{P}^n_p$-movable at ∞.

(3.6) Theorem. *Let M be an LC^n locally compact space. Then M is strongly $\mathcal{B}^n_p$-movable at ∞ if and only if M is strongly $\mathcal{P}^n_p$-movable at ∞.*

Proof. A routine proof by induction is left to the reader.

4. $\mathcal{C}_p$-movability at infinity — theorems

In this section we shall prove three theorems describing closure properties of the class of $\mathcal{C}_p$-movable at ∞ spaces. They generalize corresponding results from [6].

(4.1) Theorem. *Let M be a (weakly) $\mathcal{C}_p$-movable at ∞ space and assume N is either an ANR or $\mathcal{C}_p$ is a class of ANR pairs. If M (homotopy dominates at ∞) proper homotopy dominates N, then N is also (weakly) $\mathcal{C}_p$-movable at ∞.*

Proof. We shall deal only with the more complicated case of weak $\mathcal{C}_p$-movability at ∞. Let $f: M \to N$ and $g: N \to M$ be proper maps such that for every compact set K in N there is a compact $K_0 \supset K$ and a homotopy $H_K: (N-K_0) \times I \to N-K$ joining the inclusion $i_{K_0,K}$ of $N-K_0$ into $N-K$ with $f \circ g|_{N-K_0}$. For a compact set A in N, pick a compact set B' in M with respect to $A' = f^{-1}(A_0)$ using the fact that M is weakly $\mathcal{C}_p$-movable at ∞, and put $B = A \cup g^{-1}(B')$. If $C \supset B$ is an arbitrary compact set in N, $C' = B' \cup f^{-1}(C_0)$ is a compact subset of M. Let $D' \supset C'$ be taken with respect to B' and C', and let $D = C_0 \cup g^{-1}(D')$.

Consider a map $h: (X, X_0) \to (N-B, N-D)$ of $(X, X_0) \in \mathcal{C}_p$ into $(N-B, N-D)$. Since the composition $g \circ h$ maps the pair (X, X_0) into $(M-B', M-D')$, by assumption, there is a homotopy $G_t': X \to M-A'$, $1 \leqslant t \leqslant 2$, with $G_1' = g \circ h$, $G_2'(X) \subset M-C'$, and $G_2'|_{X_0} = g \circ h|_{X_0}$. Define $G_t: X \to N-A_0$ as $f \circ G_t'$, $1 \leqslant t \leqslant 2$. Then $G_1 = f \circ g \circ h$, $G_2(X) \subset N-C_0$, and $G_2|_{X_0} = f \circ g\ h|_{X_0}$. Finally, let $H_t: X \to N-A$, $0 \leqslant t \leqslant 1$, be the composition $(H_A)_t \circ h$ and let $F_t: X \to N-C$, $2 \leqslant t \leqslant 3$, be an extension of the partial homotopy of G_2 over X_0 given by $(H_C)_{3-t} \circ h|_{X_0}$. The join of homotopies H, G and F gives us a homotopy $X \times [0, 3] \to N-A$ starting with h and ending with a map of X into $N-C$ extending $h|_{X_0}$. Hence N is weakly $\mathcal{C}_p$-movable at ∞.

With obvious modifications, the notion of a $\mathcal{C}$-movable end of a non-compact locally compact space defined in [6] can be generalized to the notion of a *(weakly, strongly) $\mathcal{C}_p$-movable end.* By a method used in the proof of Theorem (4.11) in [6] one easily proves the following result.

(4.2) Theorem. *A non-compact, locally compact, locally connected space M is (weakly, strongly) $\mathcal{C}_p$-movable at ∞ if and only if M has finitely many (weakly, strongly) $\mathcal{C}_p$-movable ends.*

The final theorem in the present section concerns finite "complemented" products. We first state a technical definition from [7].

A closed subset A of a space X is *strongly globally right* (*left*) *unstable in* X if for each triple (U, V, W), $W \subset V \subset U$, of neighborhoods of A in X there is a map $f:(U, V, W) \to (U-A, V-A, W-A)$ such that $i \circ f|_V \simeq id_V$ and $i \circ f \simeq id_U$ ($f \circ i|_{V-A} \simeq id_{V-A}$ and $f \circ i \simeq id_{U-A}$), where $i: U-A \hookrightarrow U$ is the inclusion.

Note that a Z-set A in the Hilbert cube Q is both right and left strongly globally unstable in Q.

(4.3) Theorem. *Suppose either*

(a) N_1 *and* N_2 *are compact ANR's, or*

(b) N_1 *and* N_2 *are compact and* $\mathcal{C}_p$ *is a class of ANR pairs. Let* $X_1 \subset N_1$ *and* $X_2 \subset N_2$ *be closed subsets. Put* $X = X_1 \times X_2$, $N = N_1 \times N_2$, $M_1 = N_1 - X_1$, $M_2 = N_2 - X_2$, *and* $M = N - X$. *Assume, further, that* X_1 and X_2 *are strongly globally right unstable in* N_1 *and* N_2, *respectively, and that* X *is strongly globally left unstable in* N. *If* M_1 *and* M_2 *are* (*weakly*) $\mathcal{C}_p$*-movable at* ∞, *then* M *is* (*weakly*) $\mathcal{C}_p$*-movable at* ∞.

Proof. Let $A \subset M$ be an arbitrary compact subset. Its complement $N-A$ is an open neighborhood of X in N. Pick open neighborhoods U_1 and U_2 of X_1 and X_2, respectively, so that $A \subset N-(U_1 \times U_2)$. Put $A_1 = N_1 - U_1$ and $A_2 = N_2 - U_2$ and select $B_1 \supset A_1$ and $B_2 \supset A_2$ using the fact that M_1 and M_2 are (weakly) $\mathcal{C}_p$-movable at ∞. Put $B = (B_1 \times N_2) \cup (N_1 \times B_2)$.

Assume C is a compact set in M containing A. Choose open neighborhoods W_1 and W_2 of X_1 and X_2, respectively, such that $C \subset N-(W_1 \times W_2)$. Put $C_1 = N_1 - W_1$ and $C_2 = N_2 - W_2$ and select corresponding compact sets $D_1 \supset B_1 \cup C_1$ and $D_2 \supset B_2 \cup C_2$ applying (weak) $\mathcal{C}_p$-movability at ∞ of M_1 and M_2. Finally, let $D = (D_1 \times N_2) \cup (N_1 \times D_2)$.

Now, consider a map $f:(K, K_0) \to (M-B, M-D)$ of a pair $(K, K_0) \in \mathcal{C}_p$ into $(M-B, M-D)$. Let $h_t:(K, K_0) \to (N_1 - B_1, N_1 - D_1)$ be a homotopy connecting $h_0 = f_1 = \pi_1 \circ f$, the projection of f onto N_1, with $h_1:(K, K_0) \to (M_1 - B_1, M_1 - D_1)$. The choice of sets B_1 and D_1 implies that there is a homotopy $g_t:(K, K_0) \to (M_1 - A_1, M_1 - C_1)$ ($g_t: K \to M_1 - A_1$) such that $g_0 = h_1$, $g_1(K) \subset M_1 - C_1$, and $g_1|_{K_0} = h_1|_{K_0}$. Superposition of homotopies h_t and g_t when crossed with $f_2 = \pi_2 \circ f$ shows that there is a homotopy $\hat{f}_t:(K, K_0) \to (U_1 \times U_2, (M_1 - C_1) \times (N_2 - B_2))$ ($\hat{f}_t: K \to U_1 \times U_2$) with the following properties.

(1) $\hat{f}_0 = f$,

(2) $\hat{f}_1(K) \subset (W_1 \times W_2) - (X_1 \times X_2)$, and

(3) $\hat{f}_1|_{K_0} = (h_1|_{K_0}) \times (f_2|_{K_0})$ and $(h_1|_{K_0}) \times (f_2|_{K_0})$ is in $(N_1 - D_1) \times (N_2 - D_2)$ homotopic to $f|_{K_0}$.

By (a) (or (b)) in (3) we can, in fact, assume $\hat{f}_1|_{K_0} = f|_{K_0}$. Since X is strongly globally left unstable in N homotopy $\hat{f}_t$ can be assumed taking place in M. To finish the proof it remains to perform the similar movement, this time of the second coordinate while keeping the first coordinate unchanged, of $\hat{f}_1$.

5. $\mathcal{C}_p$-movable at infinity ANR spaces

For ANR spaces the study of $\mathcal{C}_p$-movability at ∞ is greatly simplified because on them $\mathcal{C}_p$-movability at ∞ and strong $\mathcal{C}_p$-movability at ∞ are equivalent provided $\mathcal{C}_p$ consists of metrizable pairs (Theorem (5.1)). This equivalence will be utilized in Theorem (5.5) where we prove that the Freudenthal end-point compactification of a $\mathcal{P}_p$-movable at ∞ ANR space is an ANR space. We also show that the class of ANR's that are both tame at ∞ (see [20], [11], and [5]) and $\mathcal{P}_p$-movable at ∞ agrees with the class of finitely dominated near ∞ ANR's introduced by Chapman and Ferry [10] (Corollary (5.3)), and that $\mathcal{C}_p$-movability at ∞ of an ANR depends only on shape properties of topological pairs in $\mathcal{C}_p$ (Theorem (5.4)).

(5.1) Theorem. *Let M be an ANR and let $\mathcal{C}_p$ be a class of metrizable pairs. The space M is $\mathcal{C}_p$-movable at ∞ if and only if M is strongly $\mathcal{C}_p$-movable at ∞.*

Proof. We shall prove that a $\mathcal{C}_p$-movable at ∞ ANR space M is strongly $\mathcal{C}_p$-movable at ∞. The converse is obvious.

For compact subsets A and C, $C\supset A$, of M select compact sets $B\supset A$ and $D\supset B\cup C$ using the fact that M is $\mathcal{C}_p$-movable at ∞. Consider a pair (X, X_0) in $\mathcal{C}_p$ and a map $f:(X, X_0)\to(M-B, M-D)$. Let $f_s:(X, X_0)\to(M-A, M-C)$ $(0\leqslant s\leqslant 1)$ be a homotopy such that $f_0=f$, $f_1(X)\subset M-C$, and $f_1|_{X_0}=f|_{X_0}=f_0|_{X_0}$. Let T denote the closed subset $X\times\{0, 1\}\cup X_0\times[0, 1]$ of the product $P=X\times[0, 1]$. Define a homotopy g_t $(0\leqslant t\leqslant 1)$ on T as follows

$$g_t(x, s)=\begin{cases} f_0(x), & x\in X, \ s=0,\\ H_t(x, s), & x\in X_0,\\ K_t(x), & x\in X, \ s=1,\end{cases}$$

where $H:X_0\times I\times I\to M-C$ is an extension to all of $X_0\times I\times I$ of a partial homotopy defined as f_s on $X_0\times I\times\{0\}$ and as f_0 on $X_0\times\{0\}\times I\cup X_0\times I\times\{1\}$, while $K:X\times\{1\}\times I\to M-C$ is an extension of a partial homotopy $H_t(x, 1)$ of f_1 on X_0.

It is easy to check that the homotopy g_t is well-defined and continuous. Since $g_0=f_s$ extends to all of P it follows from the homotopy extension theorem that g_1 also extends to a map $G_1:P\times[0, 1]\to M-A$. The homotopy G_1 connects $f=f_0$ with f_1 and is pointwise fixed on X_0. Hence, M is strongly $\mathcal{C}_p$-movable at ∞.

(5.2) Theorem. *If M is a $\mathcal{P}_p$-movable at ∞ ANR space, then there is a compactum A in M and proper maps $g:M-\operatorname{int}A\to(M-\operatorname{int}A)\times[0, \infty)$ and $f:(M-\operatorname{int}A)\times[0, \infty)\to M$ such that $f\circ g$ is properly homotopic to the inclusion $M-\operatorname{int}A\hookrightarrow M$.*

Proof. By [23], (4.1), (4.2), and (5.1), without loss of generality, we can asume M is a locally finite countable simplicial complex with just one $\mathcal{P}_p$-movable end.

Let $A_0 = \varnothing \subset A_1 = A \subset A_2 \subset A_3 \subset \cdots$ be an exhausting sequence of compact subpolyhedra of M such that for every index $i > 0$ there is a homotopy $\varphi_t^i : (M - \operatorname{int} A_i) \to M - \operatorname{int} A_{i-1}$ $(0 \leq t \leq 1)$ satisfying

$$\varphi_0^i = \text{the inclusion } M - \operatorname{int} A_i \hookrightarrow M - \operatorname{int} A_{i-1},$$
$$\varphi_1^i (M - \operatorname{int} A_i) \subset M - \operatorname{int} A_{i+1}, \text{ and}$$
$$\varphi_t^i \mid M - \operatorname{int} A_{i+2} = id, \quad 0 \leq t \leq 1.$$

Define a proper map $f : (M - \operatorname{int} A) \times [0, \infty) \to M$ as follows

$$f(x, t) = \begin{cases} \varphi_{t-n}^{n+1} \circ \varphi_1^n \circ \cdots \circ \varphi_1^2 \circ \varphi_1^1 (x), & 1 \leq n < t \leq n+1, \quad x \in M - \operatorname{int} A \\ \varphi_t^1 (x), & 0 \leq t \leq 1, \quad x \in M - \operatorname{int} A. \end{cases}$$

In order to define $g : M - \operatorname{int} A \to (M - \operatorname{int} A) \times [0, \infty)$, let $r : M - \operatorname{int} A \to [0, \infty)$ be a proper map. Put $g(x) = (x, r(x))$ for $x \in M - \operatorname{int} A$. Then g is a proper map and $f \circ g$ is properly homotopic to the inclusion $M - \operatorname{int} A \hookrightarrow M$.

It is interesting to know when can we replace $M - \operatorname{int} A$ by a finite complex in (5.2) and thus get that M is finitely dominated near ∞ [10]. This can be done if and only if M is tame at ∞ [20] (see also [11] and [5]), i. e., provided that for every compact subset A of M there is a larger B such that the inclusion $M - B \hookrightarrow M - A$ factors up to a homotopy through a finite complex.

(5.3) Corollary *A tame at ∞ ANR space M is $\mathcal{P}_p$-movable at ∞ if and only if M is finitely dominated near ∞.*

Proof. If M is tame at ∞ and weakly $\mathcal{P}_p$-movable at ∞ ANR space in the proof of (5.2) we can assume that the complex $M - \operatorname{int} A$ is dominated by a finite complex K [11, Lemma 5.1]. Hence, $K \times [0, \infty)$ proper homotopy dominates $(M - \operatorname{int} A) \times [0, \infty)$ [10] so that there are proper maps $a : M - \operatorname{int} A \to K \times [0, \infty)$ and $b : K \times [0, \infty) \to M$ with $b \circ a$ properly homotopic to the inclusion of $M - \operatorname{int} A$ into M. In other words, M is finitely dominated near ∞. The other implication is immediate.

We showed in [6, Theorem (4.10)] that the question of whether an ANR is $\mathcal{C}$-movable at ∞ depends only on shape properties of spaces in the class $\mathcal{C}$. The shape theory of arbitrary topological spaces as described by Kozlowski [17] was used. His construction and our proof and a definition of a shape domination for classes of spaces also apply to pairs of spaces so that we obtain the following generalization.

(5.4) Theorem. *If an ANR space M is $\mathcal{C}_p$-movable at ∞ and the class $\mathcal{C}_p$ shape dominates the class of pairs $\mathcal{D}_p$, then M is also $\mathcal{D}_p$-movable at ∞.*

The main reason for introducing movability at infinity with respect to classes of topological pairs is the next observation.

(5.5) Theorem. *If an ANR space M is $\mathcal{P}_p$-movable at ∞, then its Freudenthal end-point compactification FM is an ANR space.*

Proof. By Edwards' theorem [8], Chapman's Triangulation theorem [8] and (4.1) we can assume M is a locally finite countable simplicial complex. By (4.2), M has finitely many ends so that, deleting the interior of a sufficiently large compact subpolyhedron A, $M-\operatorname{int} A$ decomposes into finitely many disjoint one-ended and $\mathcal{P}_p$-movable at ∞ subpolyhedra. Hence, we can assume M is a strongly $\mathcal{P}_p$-movable at ∞ one-ended locally finite countable simplicial complex.

Under these assumptions we shall prove that $M\cup\infty$ is an ANR space by showing that it is strongly contractible at the point ∞ [15], i.e., that for every open neighborhood U of ∞ in $M\cup\infty$ there is a neighborhood V which is contractible inside of U to the point ∞ by a contraction that keeps the point ∞ fixed at all levels.

Given a neighborhood U of ∞ in $M\cup\infty$, put $A_0=M-U$ and select an exhausting sequence of compact subpolyhedra $A_0\subset A_1=A\subset A_2\subset A_3\subset\cdots$ and homotopies φ_t^i as in (5.2). Let $V=(M-A)\cup\infty$ and define a contraction $H:V\times[0,1]\to U$ as follows.

$$H(v,t)=\begin{cases}\varphi_{2t}^1(v), & 0\leqslant t\leqslant 1/2,\quad v\in M-A,\\ \varphi^k_{1-k^2+k(k+1)t}\circ\varphi_1^{k-1}\circ\cdots\circ\varphi_1^1(v), & v\in M-A,\\ & (k-1)/k\leqslant t\leqslant k/(k+1),\quad k>1,\\ \infty, & v=\infty,\quad 0\leqslant t\leqslant 1,\\ \infty, & v\in V,\quad t=1.\end{cases}$$

(5.6) Corollary. *The one-point compactifications $M\cup\infty$ of an $\mathcal{P}_p$--movable at ∞ ANR space M is an ANR space.*

Proof. Since EM, the end set of FM, is a finite discrete space and $M\cup\infty\cong(FM)/EM$, the corollary follows immediately from (5.5) (see section 6 below).

6. ANR divisors

In [15] Hyman defined the class of ANR divisors and proved several theorems describing properties of spaces belonging to this class. Here we shall consider compact ANR divisors. Corollary (5.6) of the previous section implies that every pointed FANR space is an ANR divisor (Theorem (6.2)). The converse of this result is not true because in (6.3) we use an example constructed by J. Dydak for other purposes to show that the class of ANR divisors is wider than the class of pointed FANR's and includes some rather pathological spaces. On the other hand, we establish a number of results about compact ANR divisors analogous tho the corresponding results about FANR's. They improve most of Hyman's theorems.

(6.1) Definition ([15]). A compact metric space X is called an ANR *divisor* provided that the space Y/X, obtained from an ANR space Y containing X as a closed subset by shrinking X to a point, is an ANR space.

The main result of [14] implies that for X to be an ANR divisor it suffices to find only one such ANR space Y for which Y/X is an ANR. Hence, by embedding a pointed FANR as a Z-set in the Hilbert cube Q, combining (3.3), (5.1), and (5.6) we get the following.

(6.2) Theorem. *Every pointed FANR space is an ANR divisor.*

The class of ANR divisors is wider than the class of pointed FANR's as the following example, communicated to the author by J. Dydak, shows.

(6.3) Example. In [12] Dydak constructed a space $X = X_1 \cup X_2$ being an FAR such that $X_1 \cap X_2$ is a pointed FANR (in fact, a circle) and X_1 is not an FANR. By the Sum Theorem (6.7) (b) below, X_1 is an ANR divisor because X and $X_1 \cap X_2$ are ANR divisors (by (6.9) and (6.2), respectively).

In attempting to characterize ANR divisors it is helpful to look for properties of FANR's that are also possessed by ANR divisors. The next four theorems handle shape invariance, cohomology finiteness, sums, and adjunction spaces.

A compactum X *strongly shape dominates* a compactum Y, in notation $Sh_s(X) \geqslant Sh_s(Y)$, if the complement $Q - X'$ of a Z-set copy X' of X in Q proper homotopy dominates the complement $Q - Y'$ of a Z-set copy Y' of Y in Q. By [9], $Sh(X) = Sh(Y)$ implies $Sh_s(X) \geqslant Sh_s(Y)$, and by (5.3) and [10], strong shape domination and shape domination coincide on the class of pointed FANR's. Also, one easily proves that $Sh_s(X) \geqslant Sh_s(Y)$ if X homotopy dominates Y so that (6.4) below extends (4.4) in [15].

(6.4) Theorem. *If $Sh_s(X) \geqslant Sh_s(Y)$ and a compactum X is an ANR divisor, then Y is also an ANR divisor.*

Proof. Consider X and Y as Z-sets in Q and put $M = Q - X$ and $N = Q - Y$. Let $f: M \to N$ and $g: N \to M$ be proper maps with $f \circ g$ proper homotopic to id_N via proper homotopy h_t.

Let U be an open neighborhood of ∞ in $N \cup \infty$. Put $A = N - U$ and select a compactum $B \supset A$ such that $h_t(N - B) \subset N - A$, for all t. The set $A' = f^{-1}(B)$ is a compactum in M. Since $M \cup \infty$ is an ANR, there is a compactum $B' \supset A'$ such that $V' = (M \cup \infty) - B'$ is contractible rel ∞ (see the proof of (5.5)) in $U' = (M \cup \infty) - A'$. Finally, put $B^* = g^{-1}(B')$ and $V = (N \cup \infty) - B^*$. Then V is contractible rel ∞ in U, i.e., $N \cup \infty$ is strongly locally contractible at the point ∞ and therefore an ANR [15].

The Čech cohomology of the suspension of an ANR divisor is isomorphic to the Čech cohomology of an ANR space as a consequence of the following isomorphism than can be derived applying various isomorphisms in [22, Chapter VI].

(6.5) Proposition. *For a closed subset X of the Hilbert cube Q and an arbitrary abelian group G there is an isomorphism*

$$\check{\tilde{H}}^q((Q - X) \cup \infty;\ G) \cong \check{\tilde{H}}^{q-1}(X;\ G)$$

of reduced Čech cohomologies.

(6.6) Corollary. *A compactum X is an ANR divisor if and only if X has finitely many components each an ANR divisor.*

(6.7) Sum Theorem. *Let X be the union of compacta X_1 and X_2 intersecting in a compactum X_0.*

(a) *If X_0, X_1, and X_2 are ANR divisors, then X is an ANR divisor.*

(b) *If X and X_0 are ANR divisors, then both X_1 and X_2 are ANR divisors.*

Proof. Embedd A into $Q\times[-1, 1]$ so that $X_1\subset Q_1=Q\times[-1, 0]$, $X\cap Q\times\{0\}=X_0$, and $X_2\subset Q_2=Q\times[0, 1]$. Then $Y=(Q\times[-1, 1])/X$ can be regarded as the union of Q_1/X_1 and Q_2/X_2 intersecting in $Q\times\{0\}/X_0$. Hence (a) and (b) follow from corresponding properties of ANR spaces [13].

(6.8) Adjunction Theorem. *Suppose X and its closed subset A are ANR divisors. If $f: A\to Y$ is a map of A into an ANR divisor Y, then the adjunction space $X\cup_f Y$ is an ANR divisor.*

Proof. Let $Q_1\subset Q$ be a Z-set copy of Q in Q and embedd X into Q such that $X\cap Q_1=A$. Let Y be a Z-set in $Q_2\cong Q$. Since Q_2 is an absolute retract and Y is a Z-set, there is an extension $f_1: Q_1\to Q_2$ of f with $f_1^{-1}(Y)=A$. The space $Q\cup_{f_1} Q_2$ is an absolute retract [13] containing a copy of $X\cup_f Y$. But $(Q\cup_{f_1} Q_2)/(X\cup_f Y)$ can be considered as $(X/X)\cup_{f_1^*}(Q_2/Y)$ where $f_1^*: Q_1/A\to\to Q_2/Y$ is the induced map. The later space is an ANR by [13]. Hence, $X\cup_f Y$ is an ANR divisor.

(6.9) Remark. Everything we said about compact ANR divisors holds also for compact AR divisors (defined analogously). Hyman [15] proved that FAR's are AR divisors (this also follows from the remark following the proof of (3.5), (5.1) and (5.6)), but the converse is an open question. It follows from (6.5) that a compactum X is an AR divisor if and only if X is an acyclic ANR divisor.

7. FANR$_n$ spaces and property UVW^n

This final section presents an application of results in §§ 3 and 4. It treats Armentrout's property UVW^n and shows that compacta having that property are n-dimensional approximation of pointed fundamental absolute neighborhood retracts.

(7.1) Definition. ([1]). A compactum X has *property UVW^n in a space N*, in notation $X\in UVW^n(N)$, if and only if for each open neighborhood U of X in N, there is an open neighorhood V of X such that for each open neighborhood W of X, there is an open neighorhood Z of X such that for each k with $0\leqslant k\leqslant n$ and each map of pairs $f:(B^k, S^{k-1})\to(V, Z)$ there is a homotopy $H:(B^k\times I, S^{k-1}\times I)\to(U, W)$ such that $(H_1(B^k), H_1(S^{k-1}))\subset(W, Z)$.

By a technique of [6] one routinely proves.

(7.2) Theorem. *A Z-set X in an ANR N has property UVW^n in N if and only if $M=N-X$ is $\mathscr{B}_p^n$-movable at ∞.*

It is equally easy to establish that statements $X\in UVW^n(N)$ and $X\times\{0\}\in\in UVW^n(N\times Q)$ are equivalent. Hence, applying a method of glueing the product $N\times[0, 1]$ of a Q-manifold N and the unit segment $[0, 1]$ with the Hilbert cube Q along $N\times\{1\}$ (see [11] and [6]), it follows from (7.2) that $X\in UVW^n(N)$ for some embedding of X into an ANR N implies that condition (7.1) holds for every ANR space N and every embedding of X into it.

We shall say that X is an *UVW^n-compactum* provided $X\in UVW^n(N)$ for every ANR space N containing X as a closed subset.

Combining theorems (5.1), (3.6), and (7.2) we see that X is an UVW^n-compactum if and only if X is an $FANR_n$ space. Here by a (*weak*) $FANR_n$ *space* we mean a compactum whose Z-set copies in the Hilbert cube Q have (weakly) $\mathcal{P}_p^n$-movable at ∞ complements. Equivalently, weak $FANR_n$'s are compacta satisfying the condition of strong movability [2] only up to dimension n.

(7.3) Examples. (a). *An* LC^{n-1} compactum is an $FANR_n$ space (see Theorem (3.4)).

(b). An approximately n-connected space is an $FANR_n$ space (see Example (3.5)).

(c). Since a weak $FANR_n$ space is n-movable, the solenoid is not a weak $FANR_1$ space.

In the special case when $\mathcal{C}_p$ is the class $\mathcal{B}_p^n$ results of § 4 together with (7.2) imply the following three theorems.

(7.4) Theorem. *If a (weak) $FANR_n$ space X (shape dominates) strongly shape dominates a compactum Y, than Y is a (weak) $FANR_n$ space.*

(7.5) Theorem. *A compactum X is a (weak) $FANR_n$ space if and only if X has finitely many components and each of them is a (weak) $FANR_n$ space.*

(7.6) Theorem. *The product $X_1 \times X_2$ is a (weak) $FANR_n$ space if and only if both X_1 and X_2 are (weak) $FANR_n$ spaces.*

The connection between this section and § 6 is provided by Armentrout's Theorem 2 in [1]. In our terminology, he proved that an $FANR_n$ space X which can be embedded in an ANR space Y such that Y/X has dimension n is an ANR divisor. In particular, every n-dimensional $FANR_{n+1}$ space X is an ANR divisor because (see [3], [18], or [11]) X embedds into an $(n+1)$-dimensional absolute retract.

The very definition of $FANR_n$ spaces together with the above observation suggest that these spaces should be FANR's when their dimension is $\leqslant n-1$. Our last theorem confirmes that suggestion.

(7.7) Theorem. *If an $FANR_{n+1}$ space X has fundamental dimension $Fd(X) \leqslant n$, then X is a pointed FANR space.*

Proof. Consider X as a Z-set in the Hilbert cube Q. We know (see [5]) that $Q-X$ is proper homotopy equivalent to Map (σ), the infinite mapping cylinder [11], where $\sigma=\{X_i, f_i\}$ is an inverse sequence of n-dimensional finite polyhedra with X_1 being contractible. Since X is an $FANR_{n+1}$ space, Map (σ) is strongly $\mathcal{P}_p^{n+1}$-movable at ∞. But, for Map (σ) this clearly implies strong $\mathcal{P}_p$-movability at ∞. Hence, $Q-X$ is strongly $\mathcal{P}_p$-movable at ∞ (see (4.1)) and, therefore by (3.3) (b), X is a pointed FANR space.

Added in the proof. Some of our results were independently obtained by Y. Kodama in his preprints "Fine movability" and "A characteristic property of a finite dimensional pointed FANR" using entirely different techniques.

The characterization of compact ANR divisors of a finite fundamental dimension was recently given by J. Dydak in the preprint entitled "On LC^n-divisors".

REFERENCES

[1] Armentrout, S., *Decompositions and absolute neighborhood retracts*, Lecture Notes in Mathematics vol. 438, Springer, New York 1975.

[2] Borsuk, K., *Theory of Shape*, Monografie Matematyczne 59, Warszawa 1975.

[3] Bothe, H., *Eine Einbettung m-dimensionalen Mengen in einen* $(m+1)$ *— dimensionalen absoluten Retrakt*, Fund. Math. 52 (1962), 209—224.

[4] Čerin, Z., *n-docility at infinity and compactifications of* LC^n *spaces*, Glasnik Mat. 12 (1977), 161—174.

[5] Čerin Z., *Locally compact spaces C-tame at infinity*, Publ. Inst. Math. 22 (1977), 49—59.

[6] Čerin, Z. *Homotopy properties of locally compact spaces at infinity — triviality and movability*, Glasnik Mat., (to appear).

[7] Čerin, Z, *Homotopy properties of locally compact spaces at infinity — calmness and smoothness*, Pacific J. Math., (to appear).

[8] Chapman, T. A., *Hilbert cube manifolds*, CBMS—NSF Regional Conferences Series, No 28, Providence 1976.

[9] Chapman, T. A., *On some applications of infinite-dimensional manifolds to the theory of shape*, Fund, Math. 76 (1972), 181—193.

[10] Chapman, T. A., and Ferry, S., *Obstruction to finiteness in the proper homotopy category*, (preprint).

[11] Chapman, T. A., and Siebenmann, L. C., *Finding a boundary for a Hilbert cube manifold*, Acta Math. 137 (1976), 171—208.

[12] Dydak, J., *Some properties of nearly* 1*-movable continua*, Bull. Acad. Polon. Sci. 25 (1977), 685—689.

[13] Hu, S.—T., *Theory of Retracts*, Wayne University Press, Detroit 1965.

[14] Hyman, D. M., *A generalization ot the Borsuk — Whitehead — Hanner theorem*, Pacific J. of Math. 23 (1967), 263—271.

[15] Hyman, D. N., *ANR divisors and absolute neighborhood contractibility*, Fund. Math. 62 (1968), 61—73.

[16] Kozlowski, G., *Factorization of certain maps up to homotopy*, Proc. Amer. Math. Soc. 21 (1969), 88—92.

[17] Kozlowski, G., *Images of ANR's* (preprint).

[18] Krasinkiewicz, J., *On a method of constructing ANR-sets. An application of inverse limits*, Fund. Math. 92 (1976), 95—112.

[19] Mardešić, S., *n-dimensional* LC^{n-1} *compacta are movable*, Bull. Acad. Polon. Sci. 19 (1971), 505—509.

[20] Siebenmann, L. C., *The obstruction to finding a boundary for an open manifold of dimension greater than five*, Thesis, Princeton University 1965. (Univ. Microfilms no. 66—5012).

[21] Siebenman, L. C., *Regular open neighborhoods*, General Topology and Appl. 3 (1973), 51—61.

[22] Spanier, E. H., *Algebraic Topology*, McGraw-Hill, New York 1966.

[23] West, J. E., *Mapping Hilbert cube manifolds to ANR*'s, (preprint).

Louisiana State University,
Department of Mathematics,
Baton Rouge, La. 70803
and
University of Zagreb,
Department of Mathematics
41001 Zagreb, pp. 187

September 1977