PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 23 (37) 1978, pp. 39—51

GRAPHS WHICH ARE SWITCHING EQUIVALENT TO THEIR LINE GRAPHS

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(Received August 9, 1977)

0. Introduction

A partition of the vertex set of a graph G into two (disjoint) subsets (one of which may be empty) will be represented as a colouring c of vertices by two colours (say black and white) such that the vertices from the same subset are coloured by the same colour. The graph G together with its colouring c will be denoted by G_c. Switching a graph G with respect to a colouring c (or partition c) means deleting all edges between black and white vertices in G_c and introducing a new edge between a black and awhite vertex whenever they were nonadjacent in G_c [6]. The graph obtained after switching will be denoted by $\mathscr{S}(G_c)$. Graphs G and H are switching equivalent if $H=\mathscr{S}(G_c)$ for some colouring c. Switching relation $\sim$ is an equivalence relation in the set of graphs and we can speak about switching classes of graphs.

In this paper we shall find all graphs which are switching equivalent to their line graphs, i. e. we shall solve the "generalized" graph equation $L(G)\sim G$. The "ordinary" graph equations were considered, for example, in [2], [3], [4], where the isomorphism relation was taken as the equality relation $=$ for graphs. Here the switching relation plays that role and normally the solutions should be switching classes of graphs. But, since the operation L is defined in the set of graphs and not in the set of switching classes, we must consider the unknown G in $L(G)\sim G$ rather as a graph than as a switching class.

The corresponding "ordinary" graph equation $L(G)=G$ was solved in [7]. The solutions are regular graphs of degree 2. All these solutions are also the solutions to $L(G)\sim G$. Let us call such solutions ordinary. As we shall see later there are some more solutions and they will be called exceptional.

Together with a solution G we shall always give a colouring c such that $L(G)=\mathscr{S}(G_c)$. The ordinary solutions can be considered as monochromatic and in the exceptional solutions G always both colours occur and, of course, we have $L(G)\neq G$.

All solutions to $L(G)\sim G$ obviously have the same number of vertices and edges. If a solution is connected it is unicyclic graph. This case will be treated in Section 1. Disconnected solutions are described in Section 2. In Section 3 we summarize the results and give some comments.

In the next, G will always denote the solution to $L(G) \sim G$.

In the proofs we primarily use Beineke's characterization of line graphs by forbidden induced subgraphs [1]. Since $L(G) = {}_c\mathcal{S}(G_c)$, G_c must not contain several graphs derived from Beineke's forbidden subgraphs by switching. In this way the following lemmas (0.1—0.4) can be easily verified.

Lemma 0.1. *If* $4K_1 \subset G$ then it is forbidden to colour three vertices of $4K_1$ by one colour and the fourth one by another. If, in addition, $C_4 \not\subset G$, then all four vertices of $4K_1$ are coloured by the same colour.*

Corollary 0.1. *If the vertices v_1, v_2, v_3, v_4 of G are mutually nonadjacent and if the vertex v_5 is not adjacent to v_1, v_2, v_3, then the vertices v_4 and v_5 have the same colour.*

Lemma 0.2. *If $K_{1,3} \subset G$, then exactly three vertices of $K_{1,3}$ are coloured by one colour and the fourth one by another. If, in addition, $C_4 \not\subset G$, then the "central" vertex of $K_{1,3}$ is coloured by one colour and the remaining vertices by another.*

Corollary 0.2. *If $K_{1,3} \cup K_1 \subset G$, then the "central" vertex in $K_{1,3}$ is coloured by one colour and all other vertices in $K_{1,3} \cup K_1$ are coloured by another colour.*

Lemma 0.3. *If $C_4 \subset G$, then the colouring in which exactly three vertices of C_4 are coloured by the same colour, is forbidden.*

Lemma 0.4. *The colourings of G, in which G_c contains as a (coloured) induced subgraph any one of the graphs of Fig. 0.1, are forbidden.*

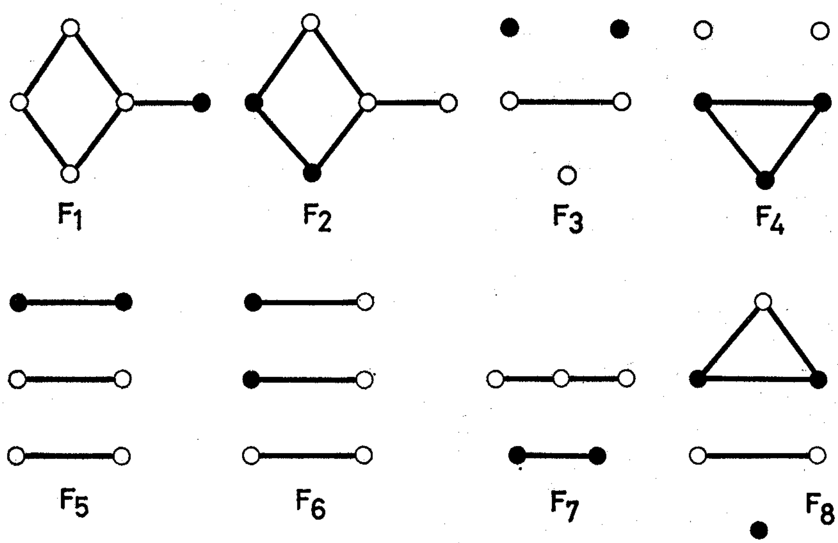

Fig 0.1.

Lemma 0.5. *If G has an odd number of vertices then the number of vertices of an odd degree is divisible by 4.*

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* $\subset$ denotes the relation "to be an induced subgraph", while $<$ denotes the relation "to be a subgraph (not necessarily induced)".

P r o o f. Let $d_1, d_2, \ldots, d_p$ be the vertex degrees of G. Then*[2] $q(L(G) = 1/2 \sum_{i=1}^{p} d_i^2 - q(G)$. If G is switched with respect to any of its vertices the parity of the number of edges is not changed. The same then holds for any switching. Since $L(G) \sim G$, $q(L(G))$ and $q(G)$ are of the same parity, $1/2 \sum_{i=1}^{p} d_i^2$ is even, and the assertion of the lemma follows immediately.

L e m m a 0.6. (*cf.* [8]) *For all graphs in the switching class of a graph with an even number of vertices, the dissection of this number into the number of vertices of an even and of an odd degree is the same.*

For all definitions and notation not defined here see [5].

1. Connected solutions

Let us consider now connected graphs G satisfying $L(G) \sim G$. As pointed out, G is then a unicyclic graph. Let us assume that G is different from a cycle, since we are interested in exceptional solutions.

L e m m a 1.1. *The maximal vertex degree in G is* 3.

P r o o f. It is sufficient to show that $K_{1.4}$ and $K_{1,4}+x$ are not induced subgraphs of G. Suppose the contrary.

1° Let $K_{1,4} \subset G$. Let v_0 be the "central" vertex of $K_{1,4}$ and v_1, v_2, v_3, v_4 its neighbours in $K_{1,4}$. According to Lemma 0.2, v_0 is coloured say, by black colour and its neighbours by the white one. Suppose first, G_c has no black vertex different from v_0. $L(G)$ is, of course, connected and hence $\mathcal{S}(G_c)$ should also be connected. That is possible only if any of the vertices v_1, v_2, v_3, v_4 has a white neighbour. Since G is unicyclic, there exist three such neighbours such that neither of them is adjacent to v_0 or mutually adjacent. But that contradicts Lemma 0.1. Thus we conclude that there exists a black vertex in G_c. different from v_0. According to Lemma 0.1 it is adjacent to two of vertices v_1, v_2, v_3, v_4 say, v_1 and v_2, and hence it must be unique. However, according to Lemma 0.2, v_1 and v_2 cannot be adjacent to white vertices. Then v_1 and v_2 are isolated in $\mathcal{S}(G_c)$ and therefore $K_{1,4} \not\subset G$.

2° Let $K_{1,4}+x \subset G$. Denote the vertices of $K_{1,4}$ as earlier by v_0, v_1, v_2, v_3, v_4 and let the edge x connect v_1 and v_2. Let v_0 be coloured black. Since $C_4 \not< G$, $v_1, v_2, v_3 v_4$ are white according to Lemma 0.2. Due to $K_4 \subset L(G)$ and $K_4 \not\subset G$, G_c contains as an induced subgraph a monochromatic triangle and an isolated vertex of another colour $(K_3 \cup K_1)$ or two isolated monochromatic edges of different colours $(2K_2)$. Since the unique triangle v_0, v_1, v_2 is not monochromatic, G_c contains together with the white edge $v_1 v_2$ also a black edge $u_1 u_2$. Consider now the three nonadjacent white vertices v_1, v_3, v_4 and one of black vertices u_1, u_2 (that which is not adjacent to the previous three vertices). This is a contradiction to Lemma 0.1 and therefore $K_{1,4}+x \not\subset G$.

This completes the proof.

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*[2] $p(H)$, $q(H)$ denote the number of vertices, edges of H, respectively.

Corollary 1.1 *G_c does not contain a monochromatic triangle and an isolated vertex of another colour as well as two isolated edges of different colours as induced subgraphs.*

Lemma 1.2. *If G contains a triangle, then there are no vertices of degree 3 outside the triangle.*

Proof. Suppose the contrary that there exists a vertex v of degree 3 outside the triangle. Since $C_4 \not< G$, acoording to Lemma 0.2, v can be considered as black and its neighbours as white. Let u be the vertex of the triangle which is the nearest to v. If* $d(u, v) \geqslant 2$, then because of Lemma 0.1 all the vertices of the triangle different from u and all the vertices of the unique path between u and v must be white. The vertex u must be black because of Corollary 1.1. If $d(u, v) \geqslant 3$, we immediately have a contradiction to Lemma 0.1. Let $d(u, v) = 2$ and let w be the vertex adjacent to both u and v. It cannot be adjacent to white vertices because of Lemma 0.2. There is no more black vertices in G_c except for u and v because of Lemma 0.1. However, $\mathcal{S}(G_c)$ is now a disconnected graph. If $d(u, v) = 1$, G_c contains one of the coloured induced subgraphs shown in Fig. 1.1. Let us discuss these three possibilities.

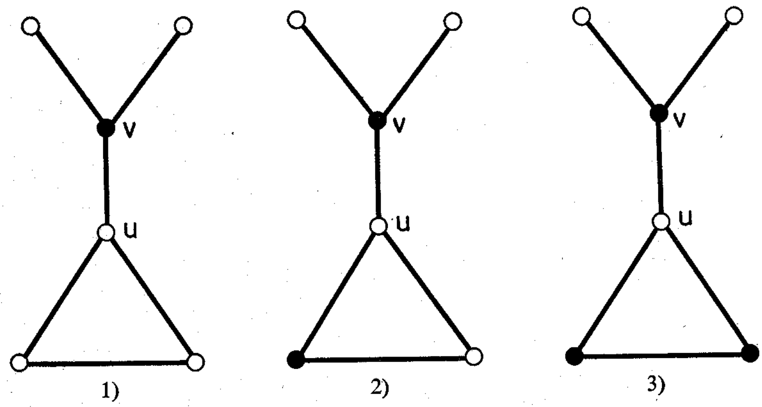


Fig. 1.1.

1° In this case G_c does not contain any black vertices more in accordance with Corollary 1.1 and Lemma 0.2. $\mathcal{S}(G_c)$ should be connected and this will be the case if the neighbours of v are adjacent to at least one white vertex. But such a configuration is impossible according to Lemma 0.1.

2° Since this subgraph is not a solution for itself we should add new vertices. Concerning vertices of the triangle, the only extension is by a new white vertex adjacent to the black vertex of the triangle (by Lemma 0.1) and this gives no solutions. By similar reasons the vertices adjacent to v can be "continued" only by one vertex and this does not give any solution.

3° Again by Lemma 0.1 and Corollary 1.1 this subgraph cannot be extended to a solution, and concerning itself it is not a solution.

This completes the proof.

* $d(u, v)$ is the distance between the vertices u and v.

According to Lemma 1.1 and 1.2, if G contains a triangle then G can be obtained from a triangle if we add to each of its vertices at most one hanging path.

Lemma 1.3. *If G is obtained from a triangle by adding three paths, then G is the graph E_1 of Fig.* 3.1.

Proof. The number of vertices of an odd degree is 6 and according to Lemma 0.5 the number of vertices $p(G)$ is even. Therefore, according to Lemma 0.6, G and $L(G)$ have the same dissections. The dissection of G is $6+(p(G)-6)$ and the dissection of $L(G)$ is $2k+(p(G)-2k)$, where k $(0\leqslant k\leqslant 3)$ is the number of paths, added to the triangle which are longer than 1. It follows that $6=2k$ or $6=p(G)-2k$ and we have the following cases: 1° $k=0$, $p(G)=6$; 2° $k=1$, $p(G)=8$; 3° $k=2$, $p(G)=10$; 4° $k=3$, $p(G)\geqslant 10$. In the first case we have the unique solution represented by the graph E_1 in Fig. 3.1. In the cases 2°, 3°, 4° we can find some possible graphs G or their subgraphs and by using Lemma 0.1 it can be easily checked that they cannot be satisfactorily coloured. This completes the proof.

Lemma 1.4. *If G is obtained from a triangle by adding two paths, then G is one of the graphs E_2, E_3, E_4 of Fig.* 3.1.

Proof. It can be easily checked that the graphs E_2, E_3, E_4 of Fig. 3.1 are the solutions. Let l_1 and l_2 be the lengths of the paths added. If $l_i\geqslant 4$ and $l_j\geqslant 1$ $(i\neq j,\ i,j=1,2)$, then according to Lemma 0.1 all the vertices of G_c are of the same colour except for $l_i=4$ and $l_j=1$. In the last case the vertex of the triangle, to which the path of length 1 is added, can be of another colour. But in this case $\mathcal{S}(G_c)$ is disconnected. For $l_i=3$ and $1\leqslant l_j\leqslant 3$ we get three graphs which are not solutions as follows by using Lemma 0.1 and the direct checking. This completes the proof.

Lemma 1.5. *If G is obtained from a triangle by adding one path, then G is one of the graphs E_5, E_6 of Fig.* 3.1.

Proof. Similarly to Lemma 1.3., the number of vertices of an odd degree is not divisible by 4 and G has an even number of vertices. The dissections for G and $L(G)$ are $2+(p(G)-2)$ and $4+(p(G)-4)$, respectively, if $G\neq K_{1,3}+x$. $G=K_{1,3}+x$ is a solution and the only possibility is then $p(G)=6$ which gives another solution. This completes the proof.

Lemma 1.6. *If the girth of G is* 4, *then G does not exist.*

Proof. Suppose the contrary. Since $G\neq C_4$ we have $C_4\cdot K_2\subset G$.* By Lemmas 0.2 and 0.4 (see F_1 of Fig. 0.1) the vertices of C_4 cannot be all coloured by the same colour. Due to Lemma 0.3 the distribution of colours $3+1$ is forbidden and hence it can be only $2+2$. If two adjacent vertices of C_4 would have the same colour then one of the forbidden colourings from Lemmas 0.2 and 0.4 would appear. Thus the vertices of C_4 are coloured in black and white alternatively (when going around the quadrangle). Now, observe

*$_4$ For the dot-product, see [5] p. 23.

the vertex outside C_4 adjacent to a vertex of C_4. By Lemma 0.2 it must be coloured different from its neighbour on C_4. Further, since $K_3 \subset L(G)$ and $K_3 \not\subset G$, G_c must contain a monochromatic edge and an isolated vertex of another colour. Clearly, the monochromatic edge cannot have vertices belonging to C_4. If the vertex of this edge which is closer to C_4 is at distance greater than 1 from the nearest vertex of C_4, then $K_4 - x \subset {}_c\mathscr{S}(G_c)$ and consequently $K_3 \subset G$ (contradiction). Thus G_c has a coloured induced subgraph represented in Fig. 1.2. This subgraph is not a solution since u and v are isolated after switching. However, u and v cannot be adjacent to some white vertices and so G_c must contain more than two black vertices, but due to Lemma 0.1 it is impossible. This proves the lemma.

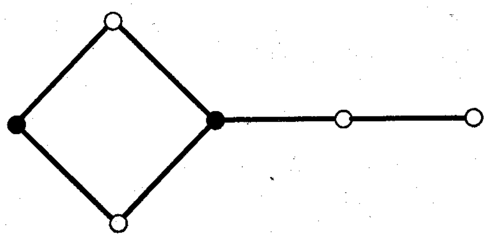


Fig. 1.2.

Lemma 1.7. *If the girth of G is greater than 4, then G does not exist.*

Proof. Suppose first that G has at least two vertices of degree 3. Let u_1, v_1 be two of them chosen so that $d(u_1, v_1)$ is as small as possible. Then the graph of Fig. 1.3 is a subgraph of G. If $2 \leqslant d(u_1, v_1) \leqslant 3$ at most one edge only between $u_i\ v_j$ $(i, j = 2, 3)$ can exist. By Lemmas 0.1 and 0.2,

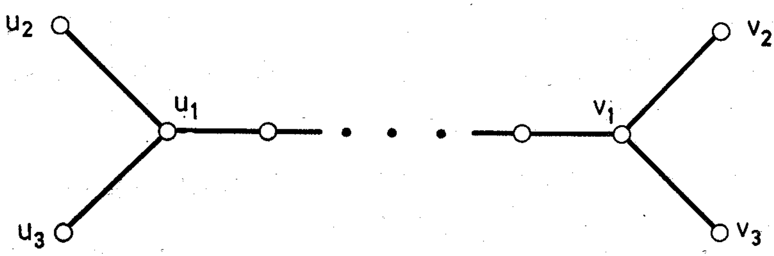


Fig. 1.3.

and Corollary 0.2, vertices u_1, v_1 are, say, black while their neighbours are white. Also it follows that $d(u_1, v_1) \neq 1$. By Lemma 0.1 all the vertices on the path $u_1 - v_1$ are white and $d(u_1, v_1) < 3$. When $d(u_1, v_1) = 2$, let w be the vertex adjacent to both u_1 and v_1. Clearly, w has no other neighbours. Due to Lemma 0.1, except for u_1, v_1 there are no other black vertices. Therefore, w is isolated in ${}_c\mathscr{S}(G_c)$. Hence, there is at most one vertex of degree 3 in G. This together with Lemma 1.1 immediately yields that G consists of a cycle with a hanging path of length l added to a vertex of the cycle. As in Lemmas 1.3 and 1.5 we can again apply Lemmas 0.5 and 0.6. Since for $l \geqslant 2$ the dissections for G and $L(G)$ are $2 + (p(G) - 2)$ and $4 + (p(G) - 4)$, respectively, we have $p(G) = 6$, a contradiction. For $l = 1$ we can easily check that such graphs are not solutions. This completes the proof.

Hence there are exactly 6 connected exceptional solutions (see graphs $E_1, \ldots, E_6$ of Fig. 3.1).

2. Disconnected solutions

Consider now disconnected exceptional solutions G of $L(G)\sim G$. The number of independent cycles of G coincides with the number of components of G.

Lemma 2.1. *G has at most three components.*

Proof. Suppose G has at least four components. Since G_c is coloured by two colours, we can choose four components and take one vertex from each of these components so that these four vertices are not of the same colour. By Lemma 0.1 two vertices are coloured by one colour and the other two by another one and there are exactly four components. By the same lemma all the vertices in each component have the same colour. Since G contains at least one edge we get F_3 from Fig. 0.1. This completes the proof.

For the next we shall take the following conveniences for the components of G_c. A component of G_c is defined to be of the type $B(W)$ if all its vertices are black (white) and it is of the type BW if it contains black as well as white vertices.

Lemma 2.2. *G cannot have three components.*

Proof. If G has three components then the following cases are possible.

1° *All components are of the type BW.* $L(G)$ has three components and $\mathcal{S}(G_c)$ is connected, a contradiction.

2° *Two components are of the type BW and the third one is of the type W (or B).* $L(G)$ has at least two components and $\mathcal{S}(G_c)$ is connected, again a contradiction.

3° *The types of components are BW, B and W.* Now $\mathcal{S}(G_c)$ is connected and $L(G)$ is connected only if the components B and W are reduced to isolated vertices. By Lemma 0.1 all the vertices of the same colour in the BW component are mutually adjacent and by Lemma 0.4 (see F_3 of Fig. 0.1) every vertex of one colour is adjacent to all vertices of another colour except perhaps for one vertex. Now, observing that G has three independent cycles we immediately get that in this case there are no solutions.

4° *One component is of the type BW and the other two are of the type W (or B).* By Lemma 0.1 all the vertices of different colours in the BW component are adjacent and the other two components are complete graphs. Now $\mathcal{S}(G_c)$ has at least two components and therefore at least one of monochromatic components should contain at least one edge. By Lemma 0.4 (observe F_3 and F_4 of Fig. 0.1) the BW component must not contain a monochromatic triangle of the black colour as well as two nonadjancent vertices of the black colour. Therefore, the BW component contains at most two vertices of the black colour and they must be adjacent. Hence*, $G=(K_i\nabla H)\cup K_m\cup K_n$ for some H and some $i\in\{1, 2\}$, where $\max(m, n)>1$. Therefrom we have

$$L(G)=L(K_i\nabla H)\cup L(L_m)\cup L(K_{n_1}) \text{ and } \mathcal{S}(G_c)=H\cup(K_i\nabla(K_m\cup K_n)).$$

Now it can be easily seen that $L^{-1}(K_i\nabla(K_m\cup K_n))$ (if it exists) is always different from $K_i\nabla H$, K_m and K_n.

*5 ∇ denotes the join of graphs [5].

5° *Two components are of the type* $W(B)$ *and the third one is of the type* $B(W)$. Now $\mathcal{S}(G_c)$ is connected and must contain two isolated vertices, i.e. $G=H\cup 2K_1$ for some H being connected. Since $\mathcal{S}(G_c)=L(G)$ we have $L(H)=K_1\nabla(K_1\cup H)$ or $2K_1\nabla H$. In the first case, since $K_{1,3}$ is a forbidden subgraph for line graphs, it follows that H is a complete graph. However, it can be easily seen that $L(K_s)\neq K_{s+1}\cdot K_2$ for any s. For the other case due to Beineke's theorem K_3 and $3K_1$ are forbidden as induced subgraph in H and now it is easy to see that there are no solutions. This complets the proof.

Now consider the case when G has two components and consequently just two independent cycles.

Lemma 2.3. *If G has two components, both of them containing an edge then G is the graph* E_7 *of Fig. 3.1.*

Proof. Let us consider the following three cases:

1° *Both components are of the type BW.* Now $L(G)$ is disconnected and has exactly two components, while $\mathcal{S}(G_c)$ has the same property only if in both components each vertex of one colour is adjacent to every vertex of another colour. Let $n_1, n_2\,(m_1, m_2)$ be the number of white (black) vertices in the first and in the second component, respectively. Since $p(\mathcal{S}(G_c))=p(L(G))$ and since G_c contains at least two adjacent vertices of the same colour (otherwise $\mathcal{S}(G_c)$ could not contain a triangle), we have $n_1+m_1+n_2+m_2\geqslant n_1m_1+n_2m_2+1$, i.e. $(n_1-1)(m_1-1)+(n_2-1)(m_2-1)\leqslant 1$. We may assume $n_1=1$. Then $n_2=m_2=2$ or $\min(n_2, m_2)=1$. By Lemma 0.1 and having in mind that the maximal vertex degree of G is greater than 2, G is one of the following graphs:

$$C_4\cup(K_{1,3}+x),\ (K_4-x)\cup K_{1,s}(s=1, 2),\ K_{1,s}\cup(2K_2\nabla K_1)\ (s=1, 2),$$
$$(K_{1,s}+x)\cup(K_{1,t}+x)\,(s, t=2, 3,\ s+t\neq 4).$$

By direct checking we find the only possibility $G=K_{1,2}\cup(K_4-X)$.

2° *One component is of the type BW and another of the type W(or B).*

Now $L(G)$ is again disconnected and has two components and $\mathcal{S}(G_c)$ should have the same property. This will be the case if the white vertices of *BW* component induce a graph having $n\geqslant 1$ components so that each vertex of only one white component is adjacent to all black vertices of G_c while the vertices of other white components (if they exist) are adjacent to at least one but not all black vertices of G_c.

Suppose $n=1$, i.e. the subgraph induced by white vertices in *BW* component is connected. Now, having in view the earlier mentioned notation, we have that $n_1+m_1+n_2\geqslant n_1m_1+n_1-1+n_2-1$, i.e. $m_1(n_1-1)\leqslant 2$ and therefrom $1\leqslant n_1\leqslant 3$. For $n_1=1$ $\mathcal{S}(G_c)$ has an isolated vertex and hence G must have an isolated edge, i.e. $m_1=1$ or $n_2=2$. In both cases the problem yields to the graph equation $L(H)=K_1\nabla H$ which according to [9] has no solutions. For $n_1=2$ we have $m_1\leqslant 2$. Now, white vertices in *BW* component induce an edge which is in $\mathcal{S}(G_c)$ an isolated edge. Clearly, *W* component must be $K_{1,2}$. So we get that $G=K_{1,2}\cup(K_4-x)$ and this is a solution already registred. For $n_1=3$ we have $m_1=1$ and *BW* component is K_4-x while the *W* component by a similar reasons as above must be P_4. By direct checking we get that $G=(K_4-x)\cup P_4$ is not a solution.

Assume now that $n>1$. Let $G_1, \ldots, G_n$ denote the components induced by white vertices in BW component of G_c and let G_1 be the component such, $K_1 \cup K_2$. that all its vertices are adjacent to all black vertices of G_c which considered themselves induce a graph G_0. We shall also denote the BW component by H_1 and the W component by H_2. By Lemma 0.1, $H_2 = K_s (s>1)$. Thus comparing $L(G)$ and $\mathcal{S}(G_c)$ we have $L(H_1) \cup L(H_2) = G_1 \cup H_3$ where H_3 is a component of $\mathcal{S}(G_c)$. The possibility $L(H_2) = H_3$ leads to the contradiction, since $K_{s+1} \subset H_3$ and $K_{s+1} \not\subset L(H_2)$. So let $L(H_2) = G_1$ and since G has just two independent cycles $G_1 = K_1$ and $H_2 = K_2$. Further, due to graphs F_5 and F_6 of Fig. 0.1 G_0 has at most one edge, and due to Lemmas 0.1 G_0 has at most three vertices. Thus G_0 is one of the following graphs $K_1 . K_2, 2K_1, K_1 \cup K_2$.

If $G_0 = K_1 (K_2)$ then $\mathcal{S}(G_c)$ contains $K_3 (K_4)$ as an induced subgraph, so that $K_3 (K_4)$ has just one common vertex (edge) with the rest of the graph. In both cases, since $L(G) = \mathcal{S}(G_c)$ it follows that G has $K_{1,3}$ as an induced, subgraph, so that just two edges of $K_{1,3}$ are pendant. Now, by using Lemma 0.2, the central vertex of $K_{1,3}$ must be black and its neighbours are white. So if G_c has only one black vertex then $\mathcal{S}(G_c)$ would have two isolated vertices a contradiction, while in the case when G_c has two adjacent black vertices the contradiction appears again due to Lemma 0.2.

If $G_0 = 2K_1$, then $\mathcal{S}(G_c)$ contains $K_4 - x$ as induced subgraph so that only the vertices of degree 2 are joined to the rest of the graph. Since $L(G) = \mathcal{S}(G_c)$, G has $K_3 . K_2$ as an induced subgraph, only its vertex of degree 1 being connected to the rest of G. The vertices of K_3 may not all be white since the graph F_3 of Fig. 0.1. would appear in G_c. It follows that the vertex of degree 3 in $K_3 \cdot K_2$ is black, others being white. So we can take $G_2 = K_2$ and G_3 must exist providing the second cycle of G. But then the graph F_6 of Fig. 0.1. appears, a contradiction.

Now let $G_0 = K_1 \cup K_2$. By the graphs F_3 and F_8 of Fig. 0.1, each (white) vertex from some $G_i (i>1)$ must be adjacent to the (black) vertex isolated in G_0. By Lemma 0.1 and since G has just two independent cycles, there are at most two such white vertices and they are adjacent. If there are just two, we get the graph F_5 of Fig. 0.1. The remaining case does not give a solution.

3° *One component is of the type W and another one of the type B.* $L(G)$ is disconnected and $\mathcal{S}(G_c)$ is connected, a contradiction.

This completes the proof.

Lemma 2.4. *If G has two components, one of which is an isolated vertex, then* $K_{1,3} \not\subset G$.

Proof. Suppose the contrary that $K_{1,3} \subset G$. Assume that the isolated vertex is white. By Corollary 0.2 the "central" vertex v_0 of $K_{1,3}$ is black and other vertices v_1, v_2, v_3 of $K_{1,3}$ are white. Suppose G_c contains one black vertex more. By Lemma 0.1 it is adjacent to at least two of vertices v_1, v_2, v_3. Therefore G_c contains one of the induced subgraphs represented in Fig. 2.1. In all three cases G_c cannot contain black vertex more because of Lemmas 0.1, 0.2 and because of the fact that G has two independent cycles. Having in view that the white vertices of degree 2 in Fig. 2.1 cannot be adjacent to some white vertices (because of Lemma 0.2), it follows that $\mathcal{S}(G_c)$ is disconnected, while $L(G)$ is connected. Hence, let G_c have only one black vertex. In order to avoid that the vertices v_1, v_2, v_3 become isolated in $\mathcal{S}(G_c)$ each of these vertices should be adjacent to a white vertex. By Lemma 0.3 there is no white vertex adjacent to two vertices of v_1, v_2, v_3 but nonadjacent to v_0.

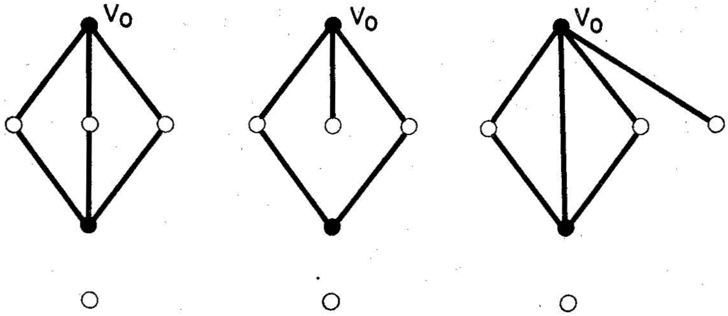


Fig. 2.1.

If it is adjacent to v_0, since $\mathcal{S}(G_c)$ is connected, we get contradiction due to Lemma 0.1. Let u_1, u_2, u_3 be the vertices adjacent to v_1, v_2, v_3 respectively. Vertices u_1, u_2, u_3 cannot induce a triangle, since then G would have to many cycles. Hence, say u_1 and u_2 are not adjacent and by Lemma 0.1 v_0 should be adjacent to at least one of them. But then the degiee of v_0 would be greater than 3 and $K_4 \subset L(G)$. Since $K_4 \not\subset G$ and since G_c contains only one black vertex, it follows that G_c contains a triangle with white vertices which are not adjacent to v_0. Now G has too many cycles or we have a colouring forbidden by Lemma 0.1. This completes the proof.

Corollary 2.1. *If G has two components one of which being an isolated vertex, then:*

1° *if two cycles of G have at most one common vertex, then they are triangles;*

2° *if two cycles of G have common edges, then we can choose two cycles in G such that they have exactly one edge in common and at least one of cycles is a triangle.*

Lemma 2.5. *If G has two components one of them being an isolated vertex and if G contains two triangles with at most one common vertex, then G is the graph E_8 of Fig. 3.1.*

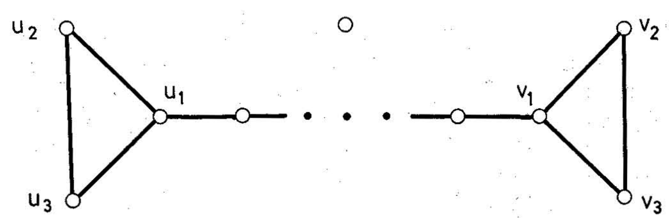
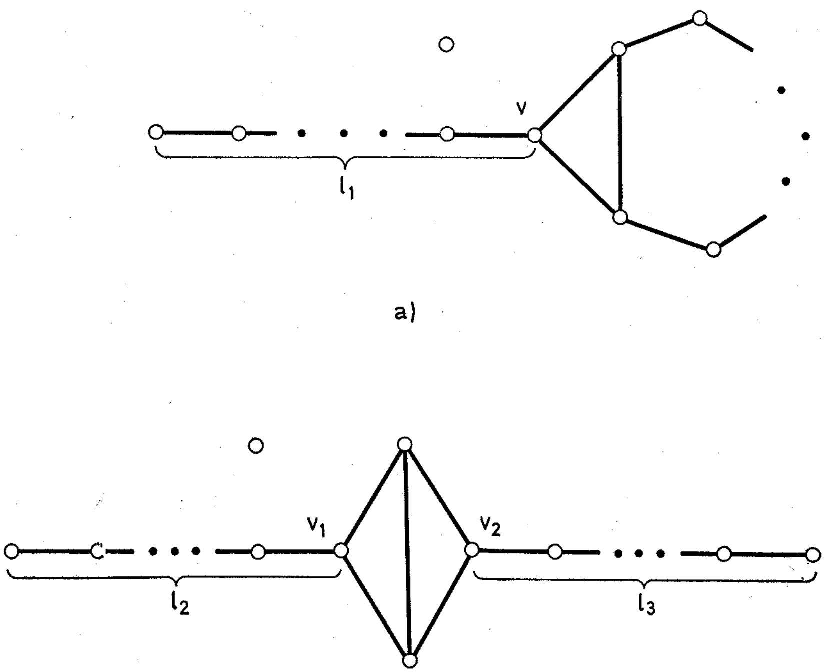


Fig. 2.2,

Proof. Now G contains an induced subgraph shown on Fig. 2.2. Let the isolated vertex be white. It can be pointed out by Lemma 0.1 that G_c does not contain any black vertex outside the subgraph of Fig. 2.2. Suppose for $d(u_1, v_1) \geqslant 1$ that a white vertex w is adjacent to one of the vertices u_2, u_3, v_2, v_3 say u_2 (to other vertices cannot be adjacent because of Lemma 2.4), By Lemma 0.1 all vertices of the subgıaph from Fig 2.2 are white except for u_2 which must be black. However, now we have $K_4 \subset {}_c\mathcal{S}(G_c)$ and* $\Delta^{*6}(G) \geqslant 4$. Since $d(u_1, v_1) \geqslant 1$, that is not possible by Lemma 2.4. It remains to check whether the graph of Fig. 2.2. is a solution for $d(u_1, v_1) \geqslant 1$. By Lemma 0.5 and 0.6 it must have eight vertices, i.e., $d(u_1, v_1) = 2$. Then by Lemma 0.2 all vertices must be white except perhaps for u_1 and v_1. Since one of them must be black ${}_c\mathcal{S}(G_c)$ will be disconnected. Let now $d(u_1, v_1) = 0$. If there exists a vertex w adjacent say, to u_2 then u_3, v_2, v_3 are white and $u_1 (= v_1)$ must be black. Since $K_4 \subset L(G)$, G_c must contain a white triangle but that is impossible having in view the the number of cycles of G. Hence, w does not exist. On the other hand, the graph of Fig. 2.2. is a solution for $d(u_1, v_1) = 0$. This completes the proof.

Lemma 2.6. *If G has two components one of which is an isolated vertex and the other one containing two cycles with a common edge, then G is one of graphs E_9, E_{10} of Fig. 3.1.*



b)

Fig. 2.3,

* $\Delta(G)$ denotes the maximal vertex degree in G.

Proof. Let the isolated vertex be white. According to Corollary 2.1. and Lemma 2.4, G has the form indicated in Fig. 2.3.[7]

1° If $C_4 \not< G$ then, by Lemma 0.1 all vertices of G_c must be white except the case when $l_1=0$ and the second cycle has a length 5. It can be easily seen that in neither case G is a solution. For $l_1 \geqslant 3$ and $C_4 < G$, by repeatedly applying Corollary 0.1 the same follows as above. Now for $l_1=1$ or 2 we get two graphs which are not solutions which can be seen by direct checking. For $l_1=0$ and $C_4<G$ we get a solution.

2° If $l_2, l_3 \geqslant 1$ applying Lemmas 0.5 and 0.6 we get that only possibilities are $(l_i, l_j) \in \{(1, 1), (1, 2), (2, 3)\}$ $i \neq j$, $i, j=2.$ 3. The possibility $l_2=l_3=1$ is a contradiction by itself while for other two by using Lemma 0.1 and its corollary we easily get that there are no solutions. If say, $l_3=0$ then by applying Corollary 0.1 for $l_2 \geqslant 3$ we get that all vertices of a triangle containing vertex v_2 are of the same colour, then v_1 and its neighbour on the hanging path are of the same colour and all vertices of the path being at distance greater than one from v_1 are also coloured by the same colour. Now G_c is monochromatic or contradicts Lemmas 0.1 or 0.4 (see F_5 in Fig. 0.1). For $l_2=2$ and $l_3=0$ we get a solution. For $l_2=1$ and $l_3=0$ the corresponding graphs are not solutions.

This completes the proof.

Hence, there are only 4 disconnected exceptional solutions (see graphs E_7, E_8, E_9, E_{10} of Fig 3.1).

3. The main result

Summarizing the result from Sections 1 and 2 we have the following theorem.

Theorem 3.1. *The only graphs which are switching equivalent to their line graphs are regular graphs of degree 2 and ten graphs shown in Fig. 3.1.*

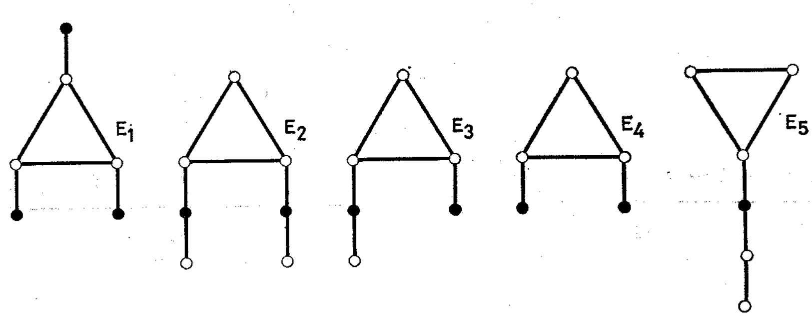


Fig. 3.1. a)

* l_1, l_2, l_3 denote the lengths of the corresponding paths.

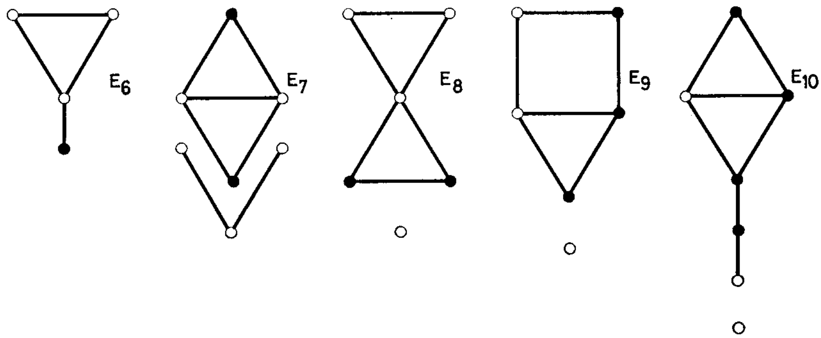


Fig 3.1. b)

For each graph in Fig. 3.1 a colouring is given which inverts G by switching into $L(G)$. In general such colourings are not unique.

The result can also be reformulated in the following way.

Theorem 3.2. *"Generalized" graph equation* $L(G) \backsim G$ *has as solutions regular graphs of degree 2 and graphs of Fig. 3.1.*

Remark. All the solutions are line graphs.

Acknowledgement We are very thankful to F.C. Bussemaker who drew our attention to a mistake in the first version of this paper.

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