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NOTE ON THE MAXIMAL SPECTRAL TYPE OF A STOCHASTIC PROCESS

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Summary. The most important problem in the time spectral analysis of a second ordered stochastic process is the problem of determining of the spectral type of a process. In this note we shall give the solution of one part of that problem; more precisely, we shall determine the maximal spectral type of a process. We shall see that, in some special cases, from our result we will obtain the maximal spectral type in terms of correlation function of the process.

1. Let $X=\{X(t),\ 0\leqslant t\leqslant 1\}$ be a purely nondeterministic stochastic process of the second order, continuous to the left in quadratic mean [2]: without loss of generality we shall assume that $EX(t)=0$ for any t. The smallest Hilbert space spanned by the random variables $X(s)$, $0\leqslant s\leqslant t$, denote by $H(X;\ t)$; put $H(X)=H(X;\ 1)$. It is known that the space $H(X)$ is separable [1]. The projection operator from $H(X)$ onto $H(X;\ t)$ denote by $E_X(t)$; it is easy to see that the family $E_X=\{E_X(t),\ 0\leqslant t\leqslant 1\}$ represents the resolution of the identity of the space $H(X)$, [2]. Each random variable $z\in H(X)$ defines the function

$$F_z(t)=\|E_X(t)\,z\|^2=E\,|E_X(t)\,z|^2,\qquad 0\leqslant t\leqslant 1,$$

which determines the measure

$$\mu_z(A)=\int_A dF_z(t),\qquad A\in\mathscr{B},$$

where $\mathscr{B}$ is the Borel σ-field over $[0;\ 1]$; the set of all measures, determined in such a way, denote by M. We say that the measure μ_y is smaller that the measure μ_z (μ_y, $\mu_z\in M$), and write $\mu_y<\mu_z$, if μ_y is absolutely continuous with respect to μ_z. We say that the measures μ_y and μ_z are equivalent, and write $\mu_y\sim\mu_z$, if $\mu_y<\mu_z$ and $\mu_z<\mu_y$. The set of all $\mu\in M$ equivalent to a given μ_z forms the equivalence class of μ_z; the elements of the set $M/\sim$ of all equivalence classes we call the spectral types. The spectral type determined by μ_z we shall denote by ρ_z, and say, also, that ρ_z is determined by z. In the set $M/_\sim$ a partial ordering is introduced in the obvious way by saying that ρ_y is smaller than ρ_z, and writing $\rho_y < \rho_z$, when the corresponding relation holds for any $\mu \in \rho_y$ and any $\bar{\mu} \in \rho_z$; we say that the spectral types ρ_y and ρ_z are equal, and write $\rho_y = \rho_z$, if $\rho_y < \rho_z$ and $\rho_z < \rho_y$. It is easy to see that the set $M/_\sim$ has the smallest (it is the spectral type identically equal to zero), and (by reason of the separability of $H(X)$) the greatest element [4]; the greatest element of $M/_\sim$ denote by ρ_X. If ρ is arbitrary spectral type smaller than ρ_X, then there is at least one element $x \in H(X)$, such that $\rho_x = \rho$, [4].

Let ρ_y and ρ_z be arbitrary elements from $M/_\sim$; the greatest spectral type which is smaller than both ρ_y and ρ_z (the smallest spectral type which is greater than both ρ_y and ρ_z) denote by $\inf\{\rho_y, \rho_z\}$ ($\sup\{\rho_y, \rho_z\}$); $\inf\{\rho_y, \rho_z\}$ and $\sup\{\rho_y, \rho_z\}$ exist for all spectral types ρ_y and ρ_z from $M/_\sim$. Also, for arbitrary ρ_z from $M/_\sim$ there is $\bar{\rho}_z$, such that $\inf\{\rho_z, \bar{\rho}_z\} = 0$ and $\sup\{\rho_z, \bar{\rho}_z\} = \rho_X$, [4]. Since the operations inf and sup are mutually distributive [4], it follows that $M/_\sim$ is the Boolean algebra. It can be shown that this Boolean algebra is complete, i.e., that infimum and supremum exist for any set of spectral types (of arbitrary cardinal number), [4].

We say that spectral types ρ_y and ρ_z are mutually orthogonal if $\inf\{\rho_y, \rho_z\} = 0$. Since the set $M/_\sim$ is bounded, it contains at most countably many mutually orthogonal and different from zero spectral types [4]. There is a set $\{\rho_1, \rho_2, \ldots\}$ of mutually orthogonal spectral types from $M/_\sim$, such that any element $\rho \in M/_\sim$ can be represented in the form $\rho = \sup\{\sigma_1, \sigma_2, \ldots\}$, where $\sigma_i < \rho_i$ for each i. We say that $\{\rho_1, \rho_2, \ldots\}$ is one basis of $M/_\sim$; the equality $\rho_X = \sup\{\rho_1, \rho_2, \ldots\}$ is satisfied. It is clear that any basis of $M/_\sim$ has at most countably many elements.

Let z be an arbitrary element from $H(X)$; the subspace $H(Z)$ of $H(X)$, defined by

$$H(Z) = \overline{\mathcal{L}}\{Z(t) = E_X(t)\, z, \quad 0 \leqslant t \leqslant 1\}$$

(where $\overline{\mathcal{L}}\{\cdot\}$ denotes the smallest Hilbert space spanned by the elements in the parantheses), we shall call the cyclic space generated by z. We say that ρ_z represents the spectral type of $H(Z)$; the spectral type of a cyclic space is uniquely determined. The multiplicity m_z of ρ_z is equal to the cardinal number of the set of all mutually orthogonal cyclic spaces of $H(X)$ of the spectral type ρ_z; we write $m_z = \text{mult}\, \rho_z$. The multiplicity of any spectral type is uniquely determined, and, by reason of the separability of $H(X)$, it cannot be greater than $\aleph_0$, [4]. It is clear that from the orthogonality of the spectral types ρ_y and ρ_z it follows the orthogonality of the cyclic spaces $H(Y)$ and $H(Z)$, [4]

2. One of the most important problem of the spectral theory of stochastic processes is the problem of determining of the maximal spectral type of arbitrary process X. Theoretically, the maximal spectral type can be determined as in Part 1. Practically, it is desirable to be able to determine ρ_X when we know only the random variables $X(t)$, $0 \leqslant t \leqslant 1$ (i.e. without the knowledge of the subspaces $H(X; t)$, $0 \leqslant t \leqslant 1$), or, much better, when we know only the correlation function of X. The next theorem gives, in some special cases, just such a possibility.

Theorem. *The maximal spectral type* ρ_X *of the stochastic process* X *satisfies the equality*

$$\rho_X = \sup_{0\leqslant t\leqslant 1} \{\rho_t\}, \tag{1}$$

where ρ_t *is the spectral type generated by* $X(t)$.

Proof. Since ρ_X is the maximal spectral type of X, it must be

$$\sup_{0\leqslant t\leqslant 1} \{\rho_t\} < \rho_X .$$

Suppose that the spectral types from this relationship are not equal, i. e., that there is a spectral type ρ_0 such that

$$\sup_{0\leqslant t\leqslant 1} \{\rho_t\} = \rho_0 < \rho_X . \tag{2}$$

Let $\{\rho_1, \rho_2, \dots\}$ be one basis of $M/_\sim$; ρ_0 can be represented in the form $\rho_0 = \sup\{\sigma_1, \sigma_2, \dots\}$, $\sigma_i < \rho_i$, $i = 1, 2, \dots$. Let us put $m_i = \operatorname{mult} \sigma_i$, $i = 1, 2, \dots$, and let H_{ij}, $j = 1, \dots, m_i$, be mutually orthogonal cyclic subspaces of $H(X)$, whose spectral types are equal to σ_i; put $H_0 = \sum_i \sum_{j=1}^{m_i} \oplus H_{ij}$. From the construction of H_0 it follows that the maximal spectral type, generated by the elements from H_0 (in respect to E_X), is equal to ρ_0. Since (by reason of (2)) $\rho_t < \rho_0$, the random variable $X(t)$ must belong to H_0 for any t, i.e., it must be $H(X) \subset H_0$. However, from (2) it follows that there exists $\rho \in M/_\sim$, such that $\inf\{\rho, \rho_0\} = 0$ and $\sup\{\rho, \rho_0\} = \rho_X$. There is an element $z \neq 0$, $z \in H(X)$, such that $\rho_z = \rho$. From $\inf\{\rho_z, \rho_0\} = 0$ it follows $\inf\{\rho_z, \sigma_i\} = 0$ for all i, which means that, in $H(X)$, there is a cyclic subspace $H(Z)$ of spectral type ρ_z, and that $H(Z)$ is orthogonal to H_0, which contradicts the previous conclusion. We proved that (1) holds. QED

Remark. If instead of the condition of the left continuity of X, we assume that $H(X; t-0) = H(X; t)$ for each t (where $H(X; t-0) = \overline{\mathscr{L}}\{X(s), 0 \leqslant s < t\}$), then the space $H(X)$ is, in general, non-separable, i.e., $\dim H(X) = \aleph_1$ (if we accept the continuum hypothesis). In this case the set $M/_\sim$ has not the maximal element — it contains continuously many mutually orthogonal spectral types. Because of that the generalized spectral types (instead of ordinary) must be considered [4]. But, in terms of these spectral types, the same result is valid.

Example 1. Let $Z = \{Z(t), 0 \leqslant t \leqslant 1\}$ be a stochastic process with orthogonal increments, such that $Z(0) = 0$ and $\|Z(t)\|^2 = E|Z(t)|^2 = F(t) < \infty$, $0 \leqslant t \leqslant 1$. Then, for arbitrary but fixed t, we have

$$F_t(s) = \| E_Z(s) Z(t) \|^2 = \begin{cases} F(s), & s \leqslant t, \\ F(t), & s > t; \end{cases}$$

hence $\mu_t(A)=\int_A dF_t(s)$, $A\in\mathcal{B}$. It is easy to see that the maximal spectral type ρ_Z of Z is determined by the measure μ_Z:

$$\mu_Z(A)=\int_A dF(s), \qquad A\in\mathcal{B}.$$

E x a m p l e 2. Let X be a wide sense Markov process, i.e., the process for which $X(t)\neq 0$, $\|X(t)\|<\infty$, $0\leqslant t\leqslant 1$, and

$$E_X(s)\,X(t)=a(t,s)\,X(s), \qquad s\leqslant t.$$

Suppose that $a(t,s)\neq 0$ for all t and s. First of all prove that the following proposition is valid:

If $\|a(t_0,s)\,X(s)\|=C(t_0)$ *for some fixed* t_0 *and any* s, $s_1\leqslant s\leqslant s_2<t_0$, *then* $\|a(t,s)\,X(s)\|=C(t)$ *for all* t *and all* s, $s_1\leqslant s\leqslant s_2$.

P r o o f. We have

$$\|a(t_0,s')\,X(s')\|=\|a(t_0,s'')\,X(s'')\| \text{ for } s_1\leqslant s',\quad s''\leqslant s_2; \tag{3}$$

put $s'<s''$. Let us show that the equality

$$a(t_0,s')\,X(s')=a(t_0,s'')\,X(s''), \qquad s_1\leqslant s'<s''\leqslant s_2 \tag{4}$$

holds. From $H(X;s')\subset H(X;s'')$ it follows

$$a(t_0,s'')\,X(s'')=a(t_0,s')\,X(s')+P_{H(X;s'')\ominus H(X;s')}\,X(t_0), \tag{5}$$

i. e.

$$\|a(t_0,s'')\,X(s'')\|=\|a(t_0,s'))\,X(s')\|+\|P_{H(X;s'')\ominus H(X;s')}\,X(t_0)\|.$$

The last term on the right side is, by reason of (3), equal to zero; hence

$$P_{H(X;s'')\ominus H(X;s')}\,X(t_0)=0,$$

which, together with (5), gives (4). From (4) we obtain

$$X(s'')=\frac{a(t_0,s')}{a(t_0,s'')}\,X(s'), \qquad s_1\leqslant s'<s''\leqslant s_2,$$

or

$$X(s)=\frac{a(t_0,s_1)}{a(t_0,s)}\,X(s_1), \qquad s_1\leqslant s\leqslant s_2. \tag{6}$$

From this equality it follows that X does not have orthogonal innovations during the interval $[s_1;s_2]$; hence

$$H(X;s_1)=H(X;s), \qquad s_1\leqslant s\leqslant s_2. \tag{7}$$

For arbitrary t, $t>s_2$, $t\neq t_0$, we have

$$P_{H(X;\,s)}X(t)=a(t,s)\,X(s),\qquad s_1\leqslant s\leqslant s_2,$$

and also

$$P_{H(X;\,s_1)}X(t)=a(t,s_1)\,X(s_1),$$

which, by reason of (7), means that

$$a(t,s_1)\,X(s_1)=a(t,s)\,X(s),\qquad s_1\leqslant s\leqslant s_2,$$

i.e.

(8) $$X(s)=\frac{a(t,s_1)}{a(t,s)}X(s_1),\qquad s_1\leqslant s\leqslant s_2.$$

From (6) and (8) we obtain

$$\frac{a(t_0,s_1)}{a(t_0,s)}=\frac{a(t,s_1)}{a(t,s)},\qquad s_1\leqslant s\leqslant s_2,\quad s_2\leqslant t,\quad s_2\leqslant t_0.$$

That means that in the interval $[s_1;s_2]$ the function $a(t,s)$ has the form

$$\left.\begin{array}{l} a(t,s)=f(t)\cdot g(s)\\ f(t)\neq 0,\quad g(s)\neq 0\end{array}\right\}\quad s_1\leqslant s\leqslant s_2<t.$$

Thus, we have

$$X(s)=\frac{g(s_1)}{g(s)}X(s_1)\quad\text{for } s_1\leqslant s\leqslant s_2,$$

which proves our proposition for $t>s_2$. It is easy to see that the proposition is also valid in the cases $t<s_1$ and $s_1\leqslant t\leqslant s_2$. QED

From this proposition it follows: If, for some t_0, the norm $\|a(t_0,s)X(s)\|$ is constant for any s, $s_1\leqslant s\leqslant s_2$, then, in the interval $[s_1;s_2]$, the norm $\|a(t,s)X(t)\|$ is constant for each t.

Now, we can determine the maximal spectral type of X. The measure μ_t is determined by the function

$$F_t(s)=\|E_X(s)X(t)\|^2=\begin{cases} a^2(t,s)\,\|X(s)\|^2 & s\leqslant t,\\ \|X(t)\|^2, & s>t.\end{cases}$$

It is easy to see that the measure μ_X, which determines ρ_X, is defined by the function

$$F(s)=a^2(1,s)\,\|X(s)\|^2,\qquad 0\leqslant s\leqslant 1.$$

Really, if the function $F(s)$ is equal to the constant on some interval $[s_1;s_2]$, then, for each t, the function $F_t(s)$ is also equal to the constant on the same interval $[s_1;s_2]$, which implies $\mu_t<\mu_X$

It is easy to see that

$$a(t,s)=\frac{r(t,s)}{r(s,s)},\qquad s\leqslant t,$$

where $r(t,s)$ is the correlation function of X: $r(t,s)=(X(t),X(s))$, $0\leqslant s,t\leqslant 1$. That is why the function $F(s)$ can be written in the form

$$F(s)=\frac{r^2(1,s)}{r(s,s)},\qquad 0\leqslant s\leqslant 1.$$

Example 3. Let $Y=\{Y(t),\ 0\leqslant t\leqslant 1\}$ be a martingal, defined on the probability space $(\Omega, \mathcal{F}, P)$; define the process X by

$$X(t)=\int_0^t \Phi(s, \omega)\, dY(s), \qquad 0\leqslant t\leqslant 1,$$

where the function $\Phi(t, \omega)$ is measurable with respect to $dt\,dP$, and, for each fixed s, $\Phi(s, \omega)$ is measurable with respect to the σ-field $\mathcal{F}_s$, which is generated by random variables $Y(u)$, $u\leqslant s$. Let

$$\int_0^1 \| \Phi(t, \omega)\|^2\, dF_*(t)<\infty,$$

where $F_*(t)=\| Y(t)\|^2$, $0\leqslant t\leqslant 1$. Then X is martingal [3] and its maximal spectral type is determined by the function

$$F(t)=\int_0^t \| \Phi(s, \omega)\|^2\, dF_*(s), \qquad 0\leqslant t\leqslant 1.$$

We obtain the same result if we put, in Example 2, $a(t, s)=1$ for all t and all s.

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