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ONE SUFFICIENT CONDITION FOR THE UNIT MULTIPLICITY OF A STOCHASTIC PROCESS

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Preliminary notions and definitions

Let $X=\{X(t),\ 0\leqslant t\leqslant 1\}$ be an arbitrary stochastic process of the second order (see [1]). Denote by $H(X;t)$ $(H(X;t-0))$ the smallest Hilbert space spanned by $X(s)$, $s\leqslant t\,(s<t)$, and suppose that $H(X;t-0)=H(X;t)$, $0\leqslant t\leqslant 1$; put $H(X)=H(X;1)$, and suppose that the space $H(X)$ is separable.

The projection-operator from $H(X)$ onto $H(X;t)$ denote by $E_X(t)$; it is easy to see that $E_X=\{E_X(t),\ 0\leqslant t\leqslant 1\}$ represents the resolution of the identity of the space $H(X)$. It is known, [1], that there exists the so-called Hida-Cramér representation of X:

$$X(t)=\sum_{n=1}^{N}\int_0^t g_n(t,u)\,dZ_n(u),\ \ 0\leqslant t\leqslant 1,$$

(1)

$$Z_n(u)=E_X(u)\,z_n;\ \ z_n\in H(X),\ \ 0\leqslant u\leqslant 1;\ \ n=1,\ldots,N.$$

The number N, which may be a positive integer or equal to infinity, we call the *multiplicity of* X.

Any element $x\in H(X)$ determines the measure m_x, which is induced by the function $F_X(t)=\|E_X(t)\,x\|^2$, $0\leqslant t\leqslant 1$. Let us put $M_X=\{m_x,\ x\in H(X)\}$. We introduce a partial ordering in M_X by saying that m_X is *subordinated* to m_y, and writing $m_x<m_y$, whenever m_x is absolutely continuous with respect to m_y. If m_x and m_y are mutually absolutely continuous, we say that they are *equivalent*, and we write $m_x\backsim m_y$. The *spectral type* μ_x is equal to the set of all elements $m\in M_X$ which are equivalent to m_x; in the set $M_X/\backsim$ a partial ordering is introduced in the obvious way: we say that μ_x is *subordinated* (*equal*) to μ_y, and we write $\mu_x<\mu_y$ $(\mu_x=\mu_y)$, if the corresponding relation holds for each $m\in\mu_x$ and each $n\in\mu_y$. For arbitrary μ_x, μ_y, by $\inf\{\mu_x,\mu_y\}$ $(\sup\{\mu_x,\mu_y\})$ we denote the greatest spectral type which is subordinated to μ_x and μ_y (the smallest spectral type to which μ_x and μ_y are subordinated). It can be shown that, under our conditions, infimum and supremum of arbitrary many spectral types exist.

Each process Z_n, $n=1,\dots,N$, from (1) determines the measure m_{z_n} and the spectral type μ_{z_n}. Note that the inequalities $\mu_{z_1}>\cdots>\mu_{z_N}$ hold, [1]; the sequence $\mu_{z_1},\dots,\mu_{z_N}$ we call the *spectral type of the process X*.

If x is arbitrary element from $H(X)$, then the smallest Hilbert space $\mathfrak{M}_x$, spanned by the variables $E_X(t)x$, $0\leqslant t\leqslant 1$, we call the *cyclic space generated by* x: $\mathfrak{M}_x=\overline{\mathscr{L}}\{E_X(t)x,\ 0\leqslant t\leqslant 1\}$; the *spectral type of* $\mathfrak{M}_x$ is equal to μ_x. The cyclic spaces, with mutually orthogonal spectral types, are mutually orthogonal. For any $\mu<\mu_x$, there is an element $y\in\mathfrak{M}_x$ such that $\mu_y=\mu$, [2].

The multiplicity of the spectral type μ_x, which we denote by mult μ_x, is equal to the cardinal number of the maximal set of mutually orthogonal cyclic spaces, whose spectral types are equal to μ_x (see [2]).

Results

Let t_0 be arbitrary but fixed value of t. The spectral type generated by $X(t)$, $0\leqslant t\leqslant 1$, we shall denote by μ_t. Put

$$\sigma_t=\inf\{\mu_t,\ \mu_{t_0}\},$$

and

$$\mathfrak{M}_t=\overline{\mathscr{L}}\{E_X(s)X(t),\ 0\leqslant s\leqslant 1\};$$

the space $\mathfrak{M}_t$ is cyclic and its spectral type is μ_t. There exists a cyclic space $\mathfrak{N}_t$, $\mathfrak{N}_t\subset\mathfrak{M}_t$, such that σ_t is its spectral type; such space $\mathfrak{N}_t$ is uniquely detemined by σ_t, [2]. It is easy to show that the projection of $\mathfrak{N}_t$ onto $\mathfrak{M}_{t_0}$ is the cyclic space; from that it follows that the space $\mathfrak{N}_t\ominus P_{\mathfrak{m}_{t_0}}\mathfrak{N}_t$ is also cyclic. There is an element $n_t\in\mathfrak{N}_t$ such that $\mathfrak{M}_{n_t}=\mathfrak{N}_t\ominus P_{\mathfrak{m}_{t_0}}\mathfrak{N}_t$. The family of all such elements n_t denote by $M_{t_0}: M_{t_0}=\{n_t,\ 0\leqslant t\leqslant 1\}$.

Theorem 1. *For arbitrary* t_0, *if*

$$\sup_{n_t\in M_{t_0}}\{\mu_{n_t}\}\lneqq\mu_{t_0},$$

then mult $\mu_{t_0}=1$.

Proof. Le us suppose that the theorem does not hold, namely that

$$\text{mult }\mu_{t_0}>1. \tag{2}$$

We shall show that this is impossible. From (2) it follows that there exist at least two mutually orthogonal cyclic spaces whose spectral types are equal to μ_{t_0}; one of these cyclic spaces, orthogonal to $\mathfrak{M}_{t_0}$, denote by ${}_1\mathfrak{W}_{t_0}$. If n_t is orthogonal to ${}_1\mathfrak{M}_{t_0}$ for each $n_t\in M_{t_0}$, then $X(t)$ is orthogonal to ${}_1\mathfrak{M}_{t_0}$ for each t, or, equivalently, $H(X)$ is orthogonal to ${}_1\mathfrak{M}_{t_0}$, which is impossible. So, there exists an element $n_{t_1}\in M_{t_0}$ such that $P_{{}_1\mathfrak{M}_{t_0}}n_{t_1}={}_1n_{t_1}\neq 0$.

The space $H(X)$ can be written in the form

$$H(X)={}_1\mathfrak{M}_{t_0}\oplus[H(X)\ominus{}_1\mathfrak{M}_{t_0}]; \tag{3}$$

since ${}_1n_{t_1}$ generates the cyclic space ${}_1\mathfrak{M}_{t_1}$ of the spectral type $\mu_{{}_1n_{t_1}}<\mu_{n_{t_1}}$ from (3) we obtain

$$H(X)={}_1\mathfrak{M}_{t_1}\oplus[{}_1\mathfrak{M}_{t_0}\ominus{}_1\mathfrak{M}_{t_1}]\oplus[H(X)\ominus{}_1\mathfrak{M}_{t_0}].$$

If we apply infinitely this reasoning we obtain

$$H(X)=\sum_{i=1}^{\infty}\oplus{}_i\mathfrak{M}_{t_i}\oplus\left[{}_1\mathfrak{M}_{t_0}\ominus\sum_{i=1}^{\infty}\oplus{}_i\mathfrak{M}_{t_i}\right]\oplus[H(X)\ominus{}_1\mathfrak{M}_{t_0}]. \tag{4}$$

It is easy to see that there is a sequence $t_1, t_2, \ldots$ such that

$${}_1\mathfrak{M}_{t_0}=\sum_{i=1}^{\infty}\oplus{}_i\mathfrak{M}_{t_i}; \tag{5}$$

namely, if it is not valid, then, for each sequence $t_1, t_2, \ldots,$ the cyclic space ${}_1\mathfrak{W}_{t_0}\ominus\sum_{i=1}^{\infty}\oplus{}_i\mathfrak{M}_{t_i}$ is orthogonal to $H(X)$, which is impossible. So, by reason of (5), the equality (4) becomes

$$H(X)=\sum_{i=1}^{\infty}\oplus{}_i\mathfrak{M}_{t_i}\oplus[H(X)\ominus{}_1\mathfrak{M}_{t_0}],$$

where the cyclic space ${}_1\mathfrak{M}_{t_i}$, $i=1, 2, \ldots,$ is generated by ${}_in_{t_i}$, and $\mu_{{}_in_{t_i}}<\mu_{n_{t_i}}$. From the mutual orthogonality of cyclic spaces ${}_i\mathfrak{M}_{t_i}$, $i=1, 2, \ldots,$ and from (5) it follows, [2],

$$\sup\{\mu_{{}_1n_{t_1}},\ \mu_{{}_2n_{t_2}}, \ldots\}=\mu_{t_0}. \tag{6}$$

Since $\mu_{{}_in_{t_i}}<\mu_{n_{t_i}}$, $i=1, 2, \ldots,$ and $n_{t_i}\in M_{t_0}$ for each i, we have

$$\sup\{\mu_{{}_1n_{t_1}},\ \mu_{{}_2n_{t_2}}, \ldots\}<\sup\{\mu_{n_{t_1}},\ \mu_{n_{t_2}}, \ldots\}<\sup_{n_t\in M_{t_0}}\{\mu_{n_t}\};$$

hence, by reason of (6),

$$\sup_{n_t\in M_{t_0}}\{\mu_{n_t}\}>\mu_{t_0},$$

which contradicts the assumption of the theorem. Q.E.D.

Theorem 2. *If* mult $\mu_t=1$ *for each* t, *then* $N=1$.

Proof. Let us suppose that the theorem does not hold, i.e., that $N>1$. For uniqueness put $N=2$. From this assumption it follows that the process X can be represented in the form

$$X(t)=\int_0^t g_1(t, u)\,dZ_1(u)+\int_0^t g_2(t, u)\,dZ_2(u)=$$

$$=X_1(t)+X_2(t), \qquad 0\leqslant t\leqslant 1,$$

where the processes X_1 and X_2 are mutually orthogonal and their multiplicities are equal to one. The process Z_i, $i=1, 2,$ determines the spectral type μ_{z_i}, and $\mu_{z_1}>\mu_{z_2}$.

The spectral type μ_t is generated by the function $F_t(s)=\|E_X(s)X(t)\|^2$, $0\leqslant s\leqslant 1$, which can be written in the form

$$F_t(s)=\|E_X(s)X_1(t)\|^2+\|E_X(s)X_2(t)\|^2=$$
$$=F_t^{(1)}(s)+F_t^{(2)}(s),\quad 0\leqslant s\leqslant 1;$$

hence

$$\mu_t=\sup\{\mu_{1t},\ \mu_{2t}\},\tag{7}$$

where the spectral type μ_{it}, $i=1, 2$, is generated by $F_t^{(i)}(s)$. If we show that there is a t, such that

$$\mu_{1t}<\mu_{z_2},\tag{8}$$

then, by reason of (7), it will imply $\mu_t<\mu_{z_2}$, But, the last relationship is equivalent to mult $\mu_t=2$, [2], which contradicts the assumption of the theorem, or, equivalently, which means that our theorem holds.

So, we must show that there is a t, such that the relationship (8) is valid. We shall show more that than; precisely, we shall show the following: If $X_1=\{X_1(t),\ 0\leqslant t\leqslant 1\}$ is arbitrary stochastic process of the unit multiplicity and the spectral type μ_{z_1}, then for any spectral type $\mu<\mu_{z_1}$ there is a t, such that $\mu_{1t}<\mu$ (here we denote $\mu_{1t}=\mu_{X_1(t)}$).

Let Z_1 be a process from the Hida-Cramér representation of X_1; the equality $H(Z_1;t)=\mathfrak{M}_{Z_1(t)}$ is satisfied for each t. It is clear that

$$\mu_{Z_1(t_1)}<\mu_{Z_1(t_2)},\ t_1\leqslant t_2,\tag{9}$$

and that, when $t\to 0$, spectral type $\mu_{Z_1(t)}$ converges to the spectral type which is identically equal to zero. From this it follows the existence of a t_0, such that $\mu_{Z_1(t_0)}<\mu$. But, the evident equality $H(X_1;t)=\mathfrak{M}_{Z_1(t)}$, $0\leqslant t\leqslant 1$, implies that $\mu_{1t}<\mu$ for each $t\leqslant t_0$, which we wanted to prove. If we put $\mu=\mu_{Z_2}$, in the last relationship, we shall obtain (8). Q.E.D.

REFERENCES

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