PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 23 (37) 1978, pp. 229—234

SOME THEOREMS FOR MODEL THEORY OF MIXED-VALUED PREDICATE CALCULI

Gradimir Vojvodić

(Communicated March 10, 1977)

We will consider k-models for mixed-valued predicate calculi. In the paper is given correspondence between k-models for mixed-valued predicate calculi and models for classical predicate calculi. The following theorems are proved: Łoš' Theorem for k-models, Compacteness Theorem for k-models and Morley's Theorem for k-models. (As in [3], [2]).

Main characteristics of mixed-valued predicate calculi, which was introduced by H. Rasiowa in [4], is: for each predicate ρ there is $n_\rho \geqslant 2$, such that ρ is n_ρ-valued.

We assume that the reader is familiar with papers [1] — [6]. Terminology and notations are the same as in [4].

A realization R in $U \neq \varnothing$ is said to be a *k-model* of a formula $\alpha \in F$, (ord $(\alpha) \leqslant m$, $m \geqslant 2$), if $\alpha_R(v) \geqslant e_k$ for each $v \in W_U$ and $0 < k < \omega$. A realization R is a *k-model of a set* $F_1 \subset F$, if it is a k-model of each formula α in F_1. A formula α is a *k-consequence* of a set $F_1 \subset F$, if each k-model of F_1 is a k-model of α. This will be written as $F_1 \models^{k} \alpha$.

Let F° be the least set of formulas in F satisfying the following condition:

1. e_0, $e_\omega \in F^\circ$;
2. $D_i(\rho^m(\tau_1, \ldots, \tau_n)) \in F^\circ$ for each $\tau_1, \ldots, \tau_n \in T$ and $\rho^m \in \Pi^m_m$, for $0 < i < \omega$, $n > 0$, $m \geqslant 2$;
3. $D_i(p^m) \in F^\circ$ for each $p^m \in V^m_0$, $0 < i < \omega$, $m \geqslant 2$;
4. if α, $\beta \in F^\circ$, then $\alpha \cup \beta$, $\alpha \cap \beta$, $\alpha \Rightarrow \beta$, $\neg \alpha$ are in F°;
5. if $\alpha(x) \in F^\circ$ and a bound individual variable ξ does not occur in $\alpha(x)$, then $\exists \xi \alpha(\xi) \in F^\circ$ and $\forall \xi \alpha(\xi) \in F^\circ$. The formulas in F° will be said to be Boolean.

Theorem 1. *For every formula α in F°, every realization R and valuation v, $\alpha_R(v) = e_0$ or $\alpha_R(v) = e_\omega$.*

Proof: This follows from example in [4, p. 217].

Theorem 2. *For every formula* α *in* F°, $0<i<\omega$, *every realization* R *and valuation* v, $(D_i(\alpha))_R(v)=\alpha_R(v)$.

Proof. This follows from T. 1 and (p_6) in [4].

Now let us define simultaneously, by induction on the length of a formula in F mappings $f_i: F\to F^\circ$, $0<i<\omega$, as follows:

(I) $f_i(e_j)=e_\omega$ for $i\leqslant j$, $0<i<\omega$, $0\leqslant j\leqslant\omega$, $f_i(e_j)=e_0$ for $i>j$, $0<i<\omega$, $0\leqslant j\leqslant\omega$

(II) $f_i(\rho^m(\tau_1, \dots, \tau_n))=D_i(\rho^m(\tau_1, \dots, \tau_n))$ for each $\tau_1, \dots, \tau_n$ and $\rho^m\in\Pi_n^m$, $m\geqslant 2$, $0<i<\omega$;

(III) $f_i(p^m)=D_i(p^m)$ for each $p^m\in V_0^m$, $m\geqslant 2$, $0<i<\omega$;

(IV) $f_i(\alpha\cup\beta)=(f_i(\alpha)\cup f_i(\beta))$, $0<i<\omega$;

(V) $f_i(\alpha\cap\beta)=(f_i(\alpha)\cap f_i(\beta))$, $0<i<\omega$;

(VI) $f_i(\alpha\Rightarrow\beta)=(f_1(\alpha)\Rightarrow f_1(\beta)\cap\dots\cap(f_i(\alpha)\Rightarrow f_i(\beta))$, $0<i<\omega$;

(VII) $f_i(\neg\alpha)=\neg f_1(\alpha)$, $0<i<\omega$;

(VIII) $f_i(D_j(\alpha))=f_j(\alpha)$, $0<i$, $j<\omega$;

(IX) $f_i(\forall\xi\alpha(\xi))=\forall\xi f_i(\alpha(\xi))$, $0<i<\omega$;

(X) $f_i(\exists\xi\alpha(\xi))=\exists\xi f_i(\alpha(\xi))$, $0<i<\omega$.

It is easy to see that $f_i(\alpha)$,$0<i<\omega$, is a closed formula iff α is closed.

Theorem 3. *For every formula* α *in* F *and every* $0<i<\omega$, $(D_i(\alpha))_R(v)=(f_i(\alpha))_R(v)$ *by any realization* R *and valuation* v.

Proof. The proof is based on (p_0) — (p_6), T. 1.7. and T. 1.9. in [4].

With the language $\mathscr{L}$ we shall associate a formalized language $\mathscr{K}$ of classical predicate calculus. Assume that $\mathscr{K}$ has the same sets of free and bound individual variables and functors as $\mathscr{L}$ and with every n-argument predicate $\rho^m\in\Pi_n^m$, $m\geqslant 2$ in $\mathscr{L}$, $n=1, 2, \dots$, there are in $\mathscr{K}$ $m-1$ predicates $\rho^{m,i}$, $i=1, \dots, m-1$ of n argument each, and no others; and that with every $p^m\in V_0^m$, $m\geqslant 2$ in $\mathscr{L}$, there are in $\mathscr{K}$ $0<i<m$ propositional variables $p^{m,i}$, and no others. Moreover, there occur in $\mathscr{K}$ the connectives $\cup$, $\cap$, $\Rightarrow$, $\neg$, the propositional constant e_0, e_ω, the quantifiers $\forall$, E and parentheses. Let $\overline{F}$ be the set of all formulas in $\mathscr{K}$. Let $f: F^\circ\to\overline{F}$ be the mapping defined thus:

(AI) $f(e_0)=e_0$, $f(e_\omega)=e_\omega$;

(AII) $f(D_i(\rho^m(\tau_1, \dots, \tau_n))=\rho^{m,i}(\tau_1, \dots, \tau_n)$ for $0<i<m$; and for $j\geqslant m$ $f(D_j(\rho^m(\tau_1, \dots, \tau_n))=\rho^{m,m-1}(\tau_1, \dots, \tau_n)$, for every $\rho^m\in\Pi_n^m$, $\tau_1, \dots, \tau_n\in T$, and $m\geqslant 2$;

(AIII) $f(D_i(p^m)) = p^{m,i}$ for $0<i<m$; and for $j \geqslant m$ $f(D_j(p^m)) = p^{m,m-1}$, for every $p^m \in V_0^m$;

(AIV) $f(\alpha \cup \beta) = (f(\alpha) \cup f(\beta))$;

(AV) $f(\alpha \cap \beta) = (f(\alpha) \cap f(\beta))$;

(AVI) $f(\alpha \Rightarrow \beta) = (f(\alpha) \Rightarrow f(\beta))$;

(AVII) $f(\neg \alpha) = \neg f(\alpha)$;

(AVIII) $f(\forall \xi \alpha(\xi)) = \forall \xi f(\alpha(\xi))$;

(AIX) $f(\exists \xi \alpha(\xi) = \exists \xi f(\alpha(\xi))$.

Let $\mathcal{S} = (\mathcal{K}, C_{\mathcal{K}})$ be a system of the classical predicate calculus with the formalized language $\mathcal{K}$ and consider the elementary theory $\mathcal{S}(\mathcal{A}) = (\mathcal{K}, C_{\mathcal{K}}, \mathcal{A})$, where $\mathcal{A}$ is the set of formulas in $\bar{F}$ defined thus: for every n-argument $(n \geqslant 1)$ predicate $\rho^m \in \Pi_n^m$, $m \geqslant 2$ in $\mathcal{L}$, let $\mathcal{A}_\rho^m$ be the conjunction of the following formulas:

(1) $\forall \xi_1, \ldots, \forall \xi_n (\rho^{m,i}(\xi_1, \ldots, \xi_n) \Rightarrow \rho^{m,i-1}(\xi_1, \ldots, \xi_n))$, for $2 \leqslant i \leqslant m-1$.

Let $\mathcal{B}_p$ m be the conjuction of the following formulas, for every $p^m \in V_0^m$

$$(p^{m,i} \Rightarrow p^{m,i-1}), \text{ for } 2 \leqslant i \leqslant m-1. \tag{2}$$

Then $\mathcal{A}$ is the set of all formulas $\mathcal{A}\rho^m$, $\mathcal{B}p^m$, for $\rho^m \in \Pi_n^m$, $p^m \in V_0^m$ and $m \geqslant 2$.

Every realization R of $\mathcal{L}$ in $U \neq \varnothing$ assigns the following realization R_0 of $\mathcal{K}$ in U: for every functor φ, $\varphi_{R_0} = \varphi_R$; for every n-argument $(n \geqslant 1)$ predicate $\rho^{m,i}$, $m \geqslant 2$ and every valuation v:

$$\rho_{R_0}^{m,i}(\tau_1, \ldots, \tau_n)(v) = D_i(\rho_R^m(\tau_1, \ldots, \tau_n))(v) \text{ for } 0<i<m, \tag{3}$$

$$\rho_{R_0}^{m,m-1}(\tau_1, \ldots, \tau_n)(v) = D_j(\rho_R^m(\tau_1, \ldots, \tau_n))(v) \text{ for } j \geqslant m; \tag{4}$$

and for every propositional variables $p^{m,i}$, $m \geqslant 2$:

$$p_{R_0}^{m,i}(v) = D_i(p_R^m(v)) \text{ for } 0<i<m, \text{ and} \tag{5}$$

$$p_{R_0}^{m,m-1}(v) = D_j(p_R^m(v)) \text{ for } j \geqslant m. \tag{6}$$

Theorem 4. *For every realization R of $\mathcal{L}$ in $U \neq \varnothing$ the realization R_0 of $\mathcal{K}$ is a model (Boolean) of $\mathcal{T}(\mathcal{A})$ and the following equation is satisfied for each formula α in F° and each valuation v: $\alpha_R(v) = (f\alpha)_{R_0}(v)$.*

Proof. The first statement follows from (3) — (6) and the fact that $D_{i+1}(a) \leqslant D_i(a)$ for each element a in $\mathcal{P}_\omega$, and T. 1.1, T. 1.7 in [4]. The proof of the second statement is by inductive argument on the length of a formula and refers to the definition of the mapping f and to equations which hold in $\mathcal{P}_\omega$ (see [4]).

Theorem 5. *If R_0 is a model (Boolean) of $\mathcal{J}(\mathcal{A})$ in $U \neq \varnothing$, then, the equations $\varphi_R = \varphi_{R_0}$, for every functor $\varphi \in \Phi$,*

$$\rho_R^m(\tau_1, \ldots, \tau_n)(v) = (\rho_{R_0}^{m,1}(\tau_1, \ldots, \tau_n)(v) \cap e_1) \cup \cdots \cup (\rho_{R_0}^{m,m-2}(\tau_1, \ldots, \tau_n)(v) \cap e_{m-2}) \cup \rho_{R_0}^{m,m-1}(\tau_1, \ldots, \tau_n)(v)$$

and

$$p_R^m(v) = (p_{R_0}^{m,1}(v) \cap e_1) \cup \cdots \cup (p_{R_0}^{m,m-2}(v) \cap e_{m-2}) \cup p_{R_0}^{m,m-1}(v)$$

define a realization R of $\mathcal{L}$ in U such that for every valuation v:

$$D_i(\rho_R^m(\tau_1, \ldots, \tau_n)(v) = \rho_{R_0}^{m,i}(\tau_1, \ldots, \tau_n)(v) \text{ for } 0 < i < m,\ m \geqslant 2, \text{ and}$$

$$D_j(\rho_R^m(\tau_1, \ldots, \tau_n)(v) = \rho_{R_0}^{m,m-1}(\tau_1, \ldots, \tau_n)(v) \text{ for } j \geqslant m,\ m \geqslant 2, \text{ and}$$

$$D_i(p_R^m(v)) = p_{R_0}^{m,i}(v) \text{ for } 0 < i < m,\ m \geqslant 2, \text{ and}$$

$$D_j(p_R^m(v)) = p_{R_0}^{m,m-1}(v) \text{ for } j \geqslant m,\ m \geqslant 2.$$

Moreover, for every formula α in F° and every valuation v the equation $\alpha_R(v) = (f\,\alpha)_{R_0}(v)$ holds.

Proof. The proof refers to (1), (2) and to (fr)-condition in [4]. The second part follows from the first and T. 4..

Theorem 6. *For any formula α in F, and for each valuation v: R is a k-model of a formula α iff R_0 is a model (Boolean) for $ff_k(\alpha)$*, i.e.

$$(\alpha)_R(v) \geqslant e_k \text{ iff } (ff_k\,\alpha)\,R_0(v) = e_\omega,$$

if the realizations R and R_0 are defined as in T.5 *and* T.6.

Proof. The proof refers to T.3, T.4 and T. 4.1. in [4].

Theorem 7. (*Loš' Theorem on reduced product for k-models*) *If ∇ is a prime filter on $N \neq \varnothing$, for any formula α in F and for each valuation v:*

$$\{n \in N : \alpha_{R_n}(v^n) \geqslant e_k\} \in \nabla \text{ iff } (\alpha)_{\Pi R_{n/\nabla}}(v) \geqslant e_k,$$

where $\Pi R_{n/\nabla}$ *is a sign for reduced product of the family realization $\{R_n : n \in N\}$ of $\mathcal{L}$ over the prime filter ∇ on N* [*see* [4]].

Proof. $\{n \in N : \alpha_{R_n}(v^n) \geqslant e_k\} \in \nabla$ iff $\{n \in N : ff_k\,\alpha\,(R_n)_0(v^n) = e_\omega\} \in \nabla$, by T.6., iff $(ff_k\,\alpha)_{\Pi R_{n0/\nabla}}(v) = e_\omega$, by Loš' theorem for classical predicate calculus [1], iff $(\alpha)_{\Pi R_{n/\nabla}}(v) \geqslant e_k$, by T.6.

Theorem 8. (*Compactness Theorem for k-models*) $F_1 \overset{k}{\models} \alpha$ *iff there is a finite subset* F_1' *of* F_1 *such that* $F_1' \overset{k}{\models} \alpha$.

Proof. $F_1 \overset{k}{\models} \alpha$ iff $\mathcal{A} \cup \{ff_k \beta : \beta \in F_1\} \models ff_k \alpha$ by T.4, T.5. and T.6., iff $\mathcal{A} \cup \{ff_k \beta : \beta \in F_1' \subset F_1$ and F_1' is a finite$\} \models ff_k \alpha$, by the compactness theorem of the classical predicate calculus (see [1]), iff $F_1' \models \alpha$, by T.4., T.5. and T.6..

Now let R and R' be any two realizations of $\mathcal{L}$ with universes U and U', reprectively. If $g : U \to U'$ is any mapping and $v : V \cup V^{\circ} \to U \cup P_{\omega}$ is a valuation, then gv denotes the valuation, defined by $gv(x) = g(v(x))$ for every $x \in V$, i.e. $gv : V \cup V^{\circ} \to U' \cup P_{\omega}$. Let be $g : U \to U'$ a bijection. Two realizations R and R' of $\mathcal{L}$ are said to be *isomorphic* if for every valuation $v : V \cup V^{\circ} \to U \cup P_{\omega}$, every $i \leqslant \omega$, every $\varphi \in \Phi$, every $\rho^m \in \Pi_n^m$, every $p^m \in V_0^m$, and $m \geqslant 2$

$$(\varphi(x_1, \ldots, x_n) = x_0)_R(v) = e_{\omega} \text{ iff } (\varphi(x_1, \ldots, x_n) = x_0)_{R'}(gv) = e_{\omega},$$

$$\rho^m(x_1, \ldots, x_n)_R(v) = e_i \qquad \text{iff } \rho^m(x_1, \ldots, x_n)_R(gv) = e_i,$$

$$p_R^m(v) \in P_m \qquad \text{iff } p_{R'}^m(gv) \in P_m.$$

Theorem 9. *Let R and R' be any two realizations of $\mathcal{L}$ and let R_0 and R_0' be defined as in T.4. Then R and R' are isomorphic iff R_0 and R_0' are isomorphic.*

Let s be an arbitrary infinite cardinal. Let F' be a subset of F. F' is said to be (s, k)-*categorical* if each two k-models of F' with universes of power s are isomorphic. Categoricity for sets of sentences is usually defined, [1].

Theorem 10. (Morley's Theorem) F' *is* (s, k)-*categorical for some infinite cardinal* s *iff* F' *is* (s, k)-*categorical for every infinite cardinal* s.

Proof. F' is (s, k)-*categorical* for some infinite cardinal s iff ff_k F' is s-categorical for some infinite cardinal s by T.9 ., T.5, and T.6, iff ff_k F' is s-categorical for every infinite cardinal s, by Morley's Theorem of the classical predicate calculus (see [1]) iff F' is (s, k)-categorical for every infinite cardinal s, by T.9., T.5. and T.6.

Remark. Let $\mathcal{A}$ be a theory for mixed valued predicate calculi. Let R be k-model for $\mathcal{A}$. Then T.5 and T.4 implies that theory $\mathcal{A}$ is not preserved under homomorphisms. (see [6], T.3.2.4).

REFERENCES

(1) J. Bell and A. Slomson, *Models and Ultraproducts*, (North Holland) Amsterdam. 1971, IX, + 322 pp.

(2) B. Dahn, *Meta-Mathematics of some Many-valued Calculi*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., vol. 22 (1974), pp 747—750.

(3) H. Rasiowa, *The Craig interpolation theorem for m-valued predicate calculi*, ibid. vol. 20 (1972), pp. 341—346.

(4) H. Rasiowa, *Mixed-valued Predicate Calculi*, Studia Logica, 34, 1975, pp. 216—234.

(5) Z. Saloni, *Gentzen rules for m-valued logic*, Bull. Acad. Polon. Sci., Sér, Sci. Math. Astronom. Phys., vol 20 (1972) pp. 819—824.

(6) C. C. Chang and H. J. Keisler: *Model theory*, North Holland, Amsterdam, 1973.

Institut za matematiku,
PMF, 21000 Novi Sad, Yugoslavija