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ON A SYSTEM OF FUNCTIONAL EQUATIONS ON QUASIGROUPS

Janez Ušan, Zoran Stojaković

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In [5] the following problem was posed:

Find all solutions of the system of functional equations

$$(1) \qquad \bigwedge_{j\in\{2,\,3,\,\ldots\,,\,n+d\}} X_1(X_2(a_1^n),\ a_{n+1}^q)=X_{2j-1}(a_1^{j-1},\ X_{2j}(a_1^{j+n-1}),\ a_{n+j}^q),$$

where n is an integer greater than 1; d nonnegative integer;

$$X_{2j-1},\ j=1,\ 2,\ \ldots\ ,\ n+d,$$

infinitary quasigroups of type ω (introduced in [1]) or finitary quasigroups of arities greater than $2n+d-2$ (that is $q\in(N\backslash\{1,\ 2,\ \ldots\ ,\ 2n+d-2\})\cup\{\infty\}$); X_{2j}, $j=1,\ 2,\ \ldots\ ,\ n+d$, are n-ary quasigroups; and all quasigroups are defined on the same nonempty set Q.

The system (1) for $q=2n+d-1$ and $d=0$ was considered in [3], and for $q=2n+d-1$ and $d\in N$ in [4].

Theorem 1. *If quasigroups A_i, $i=1,\ 2\ ,\ldots\ ,\ 2(n+d)$, defined on the same nonempty set Q, satisfy the system*

$$(1') \qquad \bigwedge_{j\in\{2,\,3,\,\ldots\,,\,n+d\}} A_1(A_2(a_1^n),\ a_{n+1}^q)=A_{2j-1}(a_1^{j-1},\ A_{2j}(a_j^{j+n-1}),\ a_{n+j}^q),$$

where $n\in N\backslash\{1\}$, $d\subset N\cup\{0\}$, $q\in(N\backslash\{1,\ 2,\ \ldots\ ,\ 2n+d-2\})\cup\{\infty\}$, then there exist 1-loop (C infinitary of type ω if $q=\infty$, finitary of arity $q-2n-d+2$ if $q\neq\infty$) and a binary group B, both defined on Q, such that for every $j\in\{1, 2, \ldots\ , n+d\}$

$$A_{2j}(a_1^n)=T_{2j-1}^{(j)-1}\overset{n-1}{B}(T_{2j-1}^{(j)}T_{2j}^{(1)}\ a_1,\ \ldots\ ,\ T_{2n-1}^{(n)}\ T_{2n}^{(1)}a_{n-j+1},$$

$$T_3^{(2)}\ T_4^{(n)}\ a_{n-j+2},\ \ldots\ ,\ T_{2j-1}^{(j)}\ T_{2j}^{(n)}\ a_n),$$

$$A_{2j-1}(a_1^{n+d},\ a_{2n+d}^q)=C(\overset{n+d-1}{B}(T_1^{(1)}T_2^{(1)}\ a_1,\ T_3^{(2)}\ T_4^{(1)}\ a_2,\ldots$$

$$\ldots,\ T_{2(j-1)-1}^{(j-1)}\ T_{2(j-1)}^{(1)}\ a_{j-1},\ T_{2j-1}^{(j)}\ a_j,\ T_{2(j+1)-1}^{(j+1)}\ T_{2(j+1)}^{(n)}\ a_{j+1},\ldots$$

$$\ldots,\ T_{2(n+d)-1}^{(n+d)}\ T_{2(n+d)}^{(n)}\ a_{n+d}),\ a_{2n+d}^q),$$

where the translations $T_i^{(t)}$ are defined by

$$T_i^{(t)} x = A_i(\underbrace{k, k, \ldots, k}_{t-1}, x, k, \ldots), \ k \text{ fixed element from } Q, \text{ and}$$

$$\overset{n-1}{B}(a_1^n) = B(B(\ldots B(Ba_1, a_2), a_3), \ldots), a_n).$$

Proof. From

$$\widehat{j \in \{2, 3, \ldots, n+d\}} \quad A_1 A_2(a_1^n), a_{n+1}^q) = A_{2j-1}(a_1^{j-1}, A_{2j}(a_j^{j+n-1}), a_{n+j}^q)$$

putting instead of each of the variables a_{2n+d}^q one fixed element $k \in Q$, we get the system

(2)
$$\widehat{j \in \{3, 3, \ldots, n+d\}} \quad \bar{A}_1(A_2(a_1^n), a_{n+1}^{2n+d-1}) = \bar{A}_{2j-1}(a_1^{j-1}, A_{2j}(a_j^{j+n-1})\, a_{n+j}^{2n+d-1}),$$

where quasigroups $\bar{A}_{2j-1}$, $j = 1, 2, \ldots, n+d$, are defined by

$$\bar{A}_{2j-1}(a_1^{n+d}) = A_{2j-1}(a_1^{n+d}, k, k, \ldots).$$

In [4] the general solution of the system (2) was given. By the Theorem 1 from that paper it follows

(3)
$$A_{2j}(a_1^n) = \bar{T}_{2j-1}^{(j)-1} \overset{n-1}{B}(\bar{T}_{2j-1}^{(j)} T_{2j}^{(1)} a_1, \ldots, \bar{T}_{2n-1}^{(n)} T_{2n}^{(1)} a_{n-j+1}, \bar{T}_3^{(2)} T_4^{(n)} a_{n-j+2}, \ldots, \bar{T}_{2j-1}^{(j)} T_{2j}^{(n)} a_n),$$

for every $j \in \{1, 2, \ldots, n+d\}$, where B is a binary group defined on Q, and translations $T_s^{(i)}$ and $\bar{T}_s^{(i)}$ are defined by

$$T_s^{(i)} x = A_s(\underbrace{k, k, \ldots, k}_{i-1}, x, k, \ldots), \quad \bar{T}_s^{(i)} x = \bar{A}_s(\underbrace{k, k, \ldots, k}_{i-1}, x, k, \ldots).$$

Since $\bar{T}_{2j-1}^{(i)} = T_{2j-1}^{(i)}$, $i, j \in \{1, 2, \ldots, n+d\}$, in the sequel we shall write $T_{2j-1}^{(i)}$ instead of $\bar{T}_{2j-1}^{(i)}$.

From

$$A_1(A_2(a_1^n), a_{n+1}^q) = A_3(a_1, A_4(a_2^{n+1}), a_{n+2}^q),$$

putting $a_3^{2n+d-1} = \overset{2n+d-3}{k}$ we shall have

$$A_1(A_2(a_1^2, \overset{n-2}{k}), \overset{n+d-1}{k}, a_{2n+d}^q) = A_3(a_1, T_4^{(1)} a_2, \overset{n+d-2}{k}, a_{2n+d}^q),$$

and if in the preceding equation we replace A_2 by the expression obtained in (3) we get

$$A_3(a_1, T_4^{(1)} a_2, \overset{n+d-2}{k}, a_{2n+d}^q) = A_1(T_1^{(1)^{-1}} \overset{n-1}{B} (T_1^{(1)} T_2^{(1)} a_1,$$

$$T_3^{(2)} T_4^{(1)} a_2, T_5^{(3)} T_6^{(1)} k, \dots, T_{2n-1}^{(n)} T_{2n}^{(1)} k), \overset{n+d-1}{k}, a_{2n+d}^q).$$

Since $T_{2i-1}^{(i)} T_{2i}^{(n)} k = T_{2i-1}^{(i)} T_{2i}^{(1)} k = e$, for every $i \in N$,

where e is the unity of the group B (which can be proved analogously as the Lemma 3 from [3] is proved), there follows

$$A_3(a_1, T_4^{(1)} a_2, \overset{n+d-2}{k}, a_{2n+d}^q) =$$

$$= A_1(T_1^{(1)-1} B(T_1^{(1)} T_2^{(1)} a_1, T_3^{(2)} T_4^{(1)} a_2), \overset{n+d-1}{k}, a_{2n+d}^q),$$

that is

$$(5) \quad A_3(a_1^2, \overset{n+d-2}{k}, a_{2n+d}^q) = A_1(T_1^{(1)^{-1}} B(T_1^{(1)} T_2^{(1)} a_1, T_3^{(2)} a_2), \overset{n+d-1}{k}, a_{2n+d}^q).$$

If we continue this procedure, that is if we fix

$$a_{j+2}^{2n+d-1} = \overset{2n+d-j-2}{k} \quad \text{in}$$

$$A_{2j-1}(a_1^{j-1}, A_{2j}(a_j^{j+n-1}), a_{n+j}^q) = A_{2j+1}(A_1^j, A_{2j+2}(a_{j+1}^{j+n}), a_{n+j+1}^q),$$

and put instead of A_{2j} the expression obtained in (3), we shall have

$$A_{2j+1}(a_1^j, T_{2j+2}^{(1)} a_{j+1}, \overset{n+d-j-1}{k}, a_{2n+d}^q) =$$

$$= A_{2j-1}(a_1^{j-1}, T_{2j-1}^{(j)^{-1}} \overset{n-1}{B} (T_{2j-1}^{(j)} T_{2j}^{(1)} a_j, T_{2j+1}^{(j+1)} T_{2j+2}^{(1)} a_{j+1},$$

$$T_{2j+3}^{(j+2)} T_{2j+4}^{(1)} k, \dots, T_{2j-1}^{(j)} T_{2j}^{(n)} k, \overset{n+d-j}{k}, a_{2n+d}^q),$$

that is

$$(6) \qquad A_{2j+1}(a_1^j, T_{2j+2}^{(1)} a_{j+1}, \overset{n+d-j-1}{k}, a_{2n+d}^q) =$$

$$= A_{2j-1}(a_1^{j-1}, T_{2j-1}^{(j)^{-1}} B(T_{2j-1}^{(j)} T_{2j}^{(1)} a_j, T_{2j+1}^{(j+1)} T_{2j+2}^{(1)} a_{j+1}), \overset{n+d-j}{k}, a_{2n+d}^q),$$

and (6) is valid for every $j = 2, 3, \dots, n+d-1$.

So we obtained the connection between the qusigroups A_{2j-1} and A_{2j+1} for every $j = 1, 2, \dots, n+d-1$ (the equation (5) is the equation (6) for $j = 1$).

If we put in (6) for $j=2$, the expression for A_3 from (5) we shall have the connection between A_1 and A_5:

$$A_5(a_1^2,\ T_6^{(1)} a_3,\ \overset{n+d-3}{k},\ a_{2n+d}^q) = A_3(a_1,\ T_3^{(2)-1} B(T_3^{(2)} T_4^{(1)} a_2,$$

$$T_5^{(3)} T_6^{(1)} a_3),\ \overset{n+d-2}{k},\ a_{2n+d}^q) = A_1(T_1^{(1)-1}\ B(T_1^{(1)} T_2^{(1)} a_1,$$

$$B(T_3^{(2)} T_{(4)}^1 a_2,\ T_5^{(3)} T_6^{(1)} a_3)),\ \overset{n+d-1}{k},\ a_{2n+d}^q),$$

that is

$$A_5(a_1^3,\ \overset{n+d-3}{k},\ a_{2n+d}^q) =$$

$$= A_1(T_1^{(1)^{-1}}\ \overset{2}{B}(T_1^{(1)} T_2^{(1)} a_1,\ T_3^{(2)} T_4^{(1)} a_2,\ T_5^{(3)} a_3,\ \overset{n+d-1}{k},\ a_{2n+d}^q).$$

Continuing this process (and using that

(7) $$T^{(1+p)}_{2(1+p)-1}\ T^{(n)}_{2(1+p)} = T^{(n+p)}_{2(n+p)-1}\ T^{(1)}_{2(n+p)},\quad p=1, 2, \dots, d,$$

which follows from

$$A_{2(1+p)-1}(a_1^p,\ A_{2(1+p)}(a_{p+1}^{p+n}),\ a_{n+p+1}^q) =$$

$$= A_{2(p+n)-1}(a_1^{p+n-1},\ A_{2(p+n)}(a_{p+n}^{p+2n-1}),\ a_{2n+p}^q),$$

when we substitute by k all variables except a_{p+n} we finally get

$$A_{2(n+d)-1}(a_1^{n+d},\ a_{2n+d}^q) = A_1\ (T_1^{(1)^{-1}}\ \overset{n+d-1}{B}\ (T_1^{(1)} T_2^{(1)} a_1,\ T_3^{(2)} T_4^{(1)} a_2, \dots$$

$$\dots, T^{(n)}_{2n-1}\ T^{(1)}_{2n}\ a_n,\ T_3^{(2)} T_4^{(n)} a_{n+1}, \dots, T^{(d)}_{2d-1}\ T^{(n)}_{2d}\ a_{n+d-1},$$

$$T^{(n+d)}_{2(n+d)-1}\ a_{n+d}),\ \overset{n+d-1}{k},\ a_{2n+d}^q).$$

Let us put

$$A_1(T_1^{(1)-1} a_1,\ \overset{n+d-1}{k},\ a_{2n+d}^q) = C(a_1,\ a_{2n+d}^q).$$

It is not difficult to see that C is an 1-loop with unity k.
From the previous results it follows

(8) $$A_{2(n+d)-1}(a_1^{n+d},\ a_{2n+d}^q) = C(\ \overset{n+d-1}{B}\ (T_1^{(1)} T_2^{(1)} a_1,\ T_3^{(2)} T_4^{(1)} a_2, \dots$$

$$\dots, T^{(n)}_{2n-1} T^{(1)}_{2n} a_n,\ \ T_3^{(2)} T_4^{(n)} a_{n+1}, \dots, T^{(d)}_{2d-1} T^{(n)}_{2d} a_{n+d-1},\ T^{(n+d)}_{2(n+d)-1} a_{n+d}),\ a_{2n+d}^q).$$

From

$$A_{2(n+d)-1}(a_1^{n+d-1},\ A_{2(n+d)}(a_{n+d}^{2n+d-1}),\ a_{2n+d}^q) =$$

$$= A_{2(n+d)-3}(a_1^{n+d-2},\ A_{2(n+d)-2}(a_{n+d-1}^{2n+d-2}),\ a_{2n+d-1}^q),$$

fixing each of the variables a_{n+d}^{2n+d-2} by k we obtain that the quasigroups $A_{2(n+d)-1}$ and $A_{2(n+d)-3}$ are isotopic:

$$A_{2(n+d)-1}(a_1^{n+d-1},\ T_{2(n+d)}^n\, a_{2n+d-1},\ a_{2n+d}^q)= \tag{9}$$

$$=A_{2(n+d)-3}(_1^{n+d-2},\ T_{2(n+d)-2}^{(1)}\, a_{n+d-1},\ a_{2n+d-1}^q),$$

and from (8) and (9)

$$A_{2(n+d)-3}(a_1^{n+d},\ (a_1^{n+d},\ a_{2n+d}^q)=C(\overset{n+d-1}{B}\ (T_1^{(1)}\,T_2^{(1)}\,a_1,\ T_3^{(2)}\,T_4^{(1)}\,a_2,\ldots$$

$$\ldots,T_{2n-1}^{(n)}\,T_{2n}^1\,a_n,\ T_3^{(2)}\,T_4^{(n)}\,a_{n+1},\ldots,T_{2(n+d-1)}^{(n+d-1)}\,a_{n+d-1},\ T_{2(n+d)-1}^{(n+d)}\,T_{2(n+d)}^{(n)}\,a_{n+d}),\ a_{2n+d}^q).$$

In a similar way it can be proved that the quasigroups A_{2j-1} and A_{2j-3} are isotopic for every $j=2,\ldots,n+d$ and from these isotopies it follows

$$A_{2j-1}(a_1^{n+d},\ a_{2n+d}^q)=C(\overset{n+d-1}{B}\ (T_1^{(1)}\,T_2^{(1)}\,a_1, \tag{10}$$

$$T_3^{(2)}\,T_4^{(1)}\,a_2,\ldots,T_{2(j-1)-1}^{(j-1)}\,T_{2(j-1)}^{(1)}\,a_{j-1},$$

$$,\ T_{2j-1}^{(j)}\,a_j,\ T_{2(j+1)-1}^{(j+1)}\,T_{2(j+1)}^{(n)}\,a_{j+1},\ldots,\ T_{2(n+d)-1}^{(n+d)}\,T_{2(n+d)}^{(n)}\,a_{n+d}),\ a_{2n+d}^q),$$

for every $j=1,2,\ldots,n+d$.

So the theorem is pıoved.

Considering (7) we get that (3) and (10) for every $j\in\{1,2,\ldots,n+d\}$ can be written in the form

$$A_{2j}(a_1^n)=T_{2j-1}^{(j)-1}\ \overset{n-1}{B}\ (\{T_{2i-1}^{(i)}\,T_{2i}^{(1)}\,a_{i-j+1}\}_{i=j}^n,\ \{T_{2i-1}^{(i)}\,T_{2i}^{(n)}\,a_{n+(i-j)}\}_{i=2}^j), \tag{3'}$$

$$A_{2j-1}(a_1^{n+d},\ a_{2n+d}^q)= \tag{10'}$$

$$=C(\overset{n+d-1}{B}\ (\{T_{2i-1}^{(i)}\,T_{2i}^{(1)}\,a_i\}_{i=1}^{j-1},\ T_{2j-1}^{(j)}\,a_j,\{T_{2i-1}^{(i)}\,T_{2i}^{(n)}\,a_i\}_{i=j+1}^{n+d}),\ a_{2n+d}^q).$$

From the theorem 1 and by a simple check we get the following theorem.

Theorem 2. *All solutions of the system* (1) *are given by*

$$X_{2j}(a_1^n)=\beta_j^{-1}\ \overset{n-1}{B}\ (\{\alpha_i\,a_{i-j+1}\}\,a_{i=j}^{j+n-1}), \tag{11}$$

$$X_{2j-1}(a_1^{n+d},\ a_{n+d+1}^q)= \tag{12}$$

$$=C(\overset{n+d-1}{B}\ (\{\alpha_i\,a_i\}_{i=1}^{j-1},\ \beta_j\,a_j,\ \{\alpha_{i+n}\,a_{i+n}\}_{i=j}^{n+d}),\ a_{2n+d+1}^q),$$

$j\in\{1,\ldots,n+d\}$, *where* α_i, β_j *are arbitrary permutations of the set* Q, B *arbitrary binary group defined on* Q, C *arbitrary* 1—*loop* (*of arity* $q-2n-d+2$ *if* $q\neq\infty$, *infinitary of type* ω *if* $q=\infty$) *defined on* Q.

As in [5] we find that the following theorem is valid.

T h e o r e m 3. *Let* X_{2j-1} *and* X_{2j} *be expressed by* B, C *and* $\overline{B}$, $\overline{C}$ *by formulae* (11) *and* (12). *Then the groups* B *and* $\overline{B}$ *are isomorfic and* $\overline{C}(x_{x_1}^{q_1}) = C(\varphi x_1, x_2^{q_1})$, $(q_1 = q - 2n + d - 1$ *if* $q \neq \infty$, $q_1 = \infty$ *if* $q = \infty)$, φ *a permutation of* Q.

If X_{2j-1} *and* X_{2j} *are expressed by* B, C, α_i, β_j *and* $\overline{B}, C, \overline{\alpha_i}, \overline{\beta_j}$ *by formulae* (11) *and* (12) *then* $\overline{B}$ *is any of the groups principaly isotopic to* B, *and* $\alpha_i \sim_B \overline{\alpha_i}$, $\beta_j \sim_b \overline{\beta_j}$ *where* $\sim_B$ *is defined by*

$$\alpha \sim_B \beta \overset{\text{def}}{\Leftrightarrow} (\exists a \in Q)(\exists b \in Q)(\alpha(x) = B(a, B(\beta(x), b))).$$

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$$\bigwedge_{j \in \{2, \dots n+d\}} X_1 [X_2 (a_1^n), a_{n+1}^{2n+d-1}] = \dots,$$

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