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CONCEPT OF THE SPECTRAL SETS IN WACHS SPACES, I

(Definitions and basic properties)

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1. Introduction

The theory of the spectral sets of bounded operators in the complex Hilbert spaces was introduced by von Neumann in [3] (see also [4]), and developed by Foias and Nagy [1], [11], [12], [13] etc, Lebow [2] and others.

With some modifications, we can introduce this notion in bounded linear operators on (left) quaternionic Banach spaces too, especially on quaternionic Hilbert spaces (i. e. Wachs spaces), and obtain many similar and also some new properties.

The connection between quaternionic and complex spectral sets is essential, but the basic problem — whether these two classes of sets are the same or not — is unsolved: no proof and no counter-example is given here. Only in some particular cases, we can prove that they are the same.

This text does not contain the "great results". All results, definitions and properties are the modifications or similar to the usual ones, expected, and the proofs are standard. But somewhere and sometime the difficult problems arise.

2. Notations

Throughout the paper, the symbols R, C, Q denote the field of real numbers (or real line), the field of complex numbers (or complex plane), and the field of quaternions (or quaternionic space). C^+ and C^- are the upper and lower closed complex half-planes, and for $q \in C$ or $q \in Q$, $\bar{q}$ is the complex (quaternionic) conjugate of q.

If P is a subset of the complex plane, we put $P^* = \{z \in C;\ \bar{z} \in P\}$; we say that P is r-symmetric of $P = P^*$.

H will denote a quaternionic (briefly q-) Banach space, and also a q-Hilbert space (Wachs space) with a q-bilinear form $\langle x, y\rangle$, H' its dual space. The corresponding complex Banach space, or complex Hilbert space $\left(\text{with form } [x, y]_i = \frac{1}{2}(\langle x, y\rangle - \mathrm{i}\langle x, y\rangle i)\right)$ is H^s.

If H is a q-Banach space, $B = B(H)$ is the algebra of all bounded q-linear operators on H, $B^s = B(H^s)$ algebra of all bounded c-linear operators on H^s.

Since obviously $B \subseteq B^s$, we may define the spectrum $\sigma(A)$ of an operator $A \in B(H)$ as the complex spectrum of A in space H^s. Then $\rho(A) = C \setminus \sigma(A)$ is its resolvent set, and if $\lambda \in \rho(A)$, $R(A; \lambda) = (A - \lambda I)^{-1}$ is resolvent of A.

If $\lambda \in Q$, we put $K(\lambda) = \lambda I$, $\bar{K}(\lambda) = K(\bar{\lambda})$, and $K_0 = K(j)$ (i, j, k are the imaginary quaternionic units).

3. A functional calculus

Assume that H is an arbitrary q-Banach space. Here we introduce a functional calculus of an operator $A \in B(H)$, more precisely the largest one contained in the classic Riesz-Dunford's functional calculus in H^s.

L e m m a. *Algebra B is a real subalgebra of the complex algebra B^s, and the next decomposition is valid*: $B^s = B + (iB)$.

The corresponding projection $P\colon\ B^s \to B$ *is* $P(X) = \frac{1}{2}(X + K_0 X \bar{K}_0)$ $(X \in B(H^s))$.

P r o o f. Obviously, $B(H) = B$ is a closed real subalgebra of B^s, and immediately $B \cap (iB) = \{0\}$. Since we can write an arbitrary operator $X \in B^s$ in the form

$$X = \frac{1}{2}(X + K_0 X \bar{K}_0) + \frac{i}{2}\,\frac{(-i)}{2}(X - K_0 X \bar{K}_0) = X_1 + iX_2$$

it is not difficult to see that operators

$$P(X) = \frac{1}{2}(X + K_0 X \bar{K}_0), \quad P_1(X) = \frac{\bar{i}}{2}(X - K_0 X \bar{K}_0)$$

belong to $B = B(H)$. □

Suppose $A \in B(H)$. Let $\mathcal{A}^s = [A^s]$ be the class of all complex functions $f(z)$ defined on an open set $\Delta(f)$ in C, which contains the spectrum $\sigma(A)$ and $f(z)$ is locally analytic on $\Delta(f)$ (Taylor [5], p. 288).

Let $\mathcal{A}$ be the class of all functions $f \in \mathcal{A}^s$ such that $f(\bar{z}) = \overline{f(z)}$ for $z \in \Delta(f) \cap \Delta(f)^*$. We call them *r-symmetric* locally analytic functions.

If $f \in \mathcal{A}^s$ and D is any bounded Cauchy domain ($\sigma(A) \subseteq D$, $\bar{D} \subseteq \Delta(f)$), one can put:

$$(*) \qquad f(A) = -\frac{1}{2\pi i} \int_{\partial+(D)} K(f(z))\, R(A: z)\, dK(z),$$

and $f(A) \in B^s$ (Taylor [5], p. 289).

Every such Cauchy domain is called the admissible domain of f.

Lemma. 2. *If D is any admissible domain of a locally analytic function $f(z)$ on $\sigma(A)$, then $D_1 = D \cap D^*$ is such a domain too.*

Proof. For any arbitrary function $f \in \mathcal{A}^s$ locally analytic on $\sigma(A)$, there exists at least one admissible domain D ($\sigma(A) \subseteq D$, $\bar{D} \subseteq \Delta(f)$), and the integral (*) is independent of the choice of the domain D ([5], p. 280).

Since the spectrum $\sigma(A)$ is r-symmetric, it is easy to verify that together with D, $D_1 = D \cap D^*$ is also a Cauchy domain of $f \in \mathcal{A}^s$, and that $\partial(D_1)$ is homotopic with $\partial(D)$. □

That is why we can restrict us to the case when the corresponding domain D is a r-symmetric set.

Theorem 1. *Suppose $f \in \mathcal{A}^s$. Then the operator*

$$f(A) = -\frac{1}{2\pi i} \int_{\partial+(D)} K(f(z))\, R(A; z)\, dK(z)$$

(D-any r-symmetric admissible domain of f), belongs to $B(H)$ if and only if there exists an r-symmetric locally analytic function $g \in \mathcal{A}$ on $\sigma(A)$, equivalent to f.

Proof. The condition is sufficient. In order to prove that the operator $f(A) = g(A) \in B$, it is enough to prove that $K_0 g(A) \bar{K}_0 = g(A)$.

In view of the definition of the operator-integral (*), it is easy to see that the following is true:

$$K_0 g(A) \bar{K}_0 = +\frac{1}{2\pi i} \int_{\partial+(D)} K_0 K(g(z))\, R(A; z)\, d(z\bar{K}_0) =$$

$$= +\frac{1}{2\pi i} \int_{z \in \partial+(D)} K(\overline{g(z)})\, R(A; \bar{z})\, dK(\bar{z}) =$$

$$= +\frac{1}{2\pi i} \int_{z \in \partial+(D)} K(g(\bar{z}))\, R(A; \bar{z})\, dK(\bar{z}).$$

Since D is a r-symmetric domain, obviously $z \in \partial_+ (D)$ if and only if $\bar{z} \in \partial_- (D^*) = \partial_- (D)$, wherefrom we obtain:

$$K_0 g(A) \bar{K}_0 = +\frac{1}{2\pi i} \int_{\partial-(D)} K(g(z'))\, R(A; z')\, dK(z') =$$

$$= -\frac{1}{2\pi i} \int_{\partial+(D)} K(g(z))\, R(A; z)\, dK(z) =$$

$$= g(A)$$

so that $g(A)$ is in $B(H)$.

The condition is necessary. Suppose $f(A) \in B(H)$ i.e. $K_0 f(A) \bar{K}_0 = f(A)$. In the similar way as before we obtain:

$$f(A) = -\frac{1}{2\pi i} \int_{\partial+(D)} K(\overline{f(\bar{z})})\, R(A; z)\, dK(z) =$$

$$= g(A),$$

where $g(z) = \frac{1}{2}(f(z) + \overline{f(\bar{z})})$. Obviously $g(z)$ belongs to $\mathcal{A}$, q. e. d. □

In the real algebra $\mathcal{A}$ we identify the functions which are equal on an open r-symmetric set contained the spectrum $\sigma(A)$, and introduce the usual composition: $(f \circ g)(z) = f(z)\, g(z)$. Then, as in the complex case, one can prove that $f \to f(A)$ is an algebraic homomorphism of the real algebra (of equivalence classes) $\mathcal{A}$ in the real algebra $B(A)$.

The functional calculus so obtained is resonably to call — the quaternionic Riesz-Dunford functional calculus.

Let us suppose H is a Wachs space.

Proposition 1. *The classes* $\mathcal{A} = [A]$ *and* $\mathcal{A}^* = [A^*]$ *are the same. If* $f \in \mathcal{A}$ *then* $f(A^*) = f(A)^*$.

Proof. Since $\sigma(A)$ is a r-symmetric set, it is easily seen that $f \in \mathcal{A}$ implies $f \in \mathcal{A}^*$, and conversely. In this case we have

$$f(A^*) = -\frac{1}{2\pi i} \int_{\partial+(D)} K(f(z))\, R(A^*; z)\, dK(z) =$$

$$= \left\{ +\frac{1}{2\pi i} \int_{z \in \partial+(D)} K(f(\bar{z}))\, R(A; \bar{z})\, dK(\bar{z}) \right\}^* =$$

$$= \left\{ -\frac{1}{2\pi i} \int_{z \in \partial+(D)} K(f(z))\, R(A; z)\, dK(z) \right\}^* =$$

$$= f(A)^*. \quad \square$$

From Proposition 1 we conclude that for A symmetric, all operators obtained in this way are symmetric too. But for A skew-symmetric there is no analogy (if we do not restrict ourselves to some subclasses of functions). So obtained operators ary only normal.

4. Spectral sets in Wachs spaces

Although the next definition and some properties are true in more general quaternionic Banach spaces too, if the contrary is not said, we assume the space H be a Wachs space.

Definition 1. *A closed set X in the (extended) complex plane is said to be q-spectral (or spectral) set of an operator $A \in B(H)$:*

1° X contains the spectrum $\sigma(A)$;

2° For every r-symmetric rational function $f(z)$ having no poles on X, the next relation is valid:

$$\|f(A)\| \leqslant \sup_{z \in X} |f(z)|.$$ [1]

The spectral sets of the operator A as an operator on H^s are denoted by prefix "c-" (complex spectral sets).

Obviously *every c-spectral set of an operator $A \in B(H)$ is q-spectral.* But we do not know whether the converse is true, i. e. *whether q-spectral and c-spectral sets are the same or not!* We think that they must be different but we have not any counter-example.

We give one weak estimation for q-spectral sets in this sense. In view of the Proposition 3, we may restrict us ourselves the r-symmetric q-spectral sets.

Proposition 2. *Suppose an r-symmetric spectral set X of an operator A, and an arbitrary rational function $f(z)$ having no poles on X are given. Then the following is true:*

$$\|f(A)\| \leqslant 2 \sup_{z \in X} |f(z)|.$$

Proof. By $\pi(f)$ let us denote the set of all poles of the function $f(z)$.

For each $z \notin \pi(f) \cup \pi(f)^*$, we can write the function $f(z)$ in the form

$$f(z) = g(z) + ih(z), \tag{1}$$

where $g(z) = \dfrac{1}{2}(f(z) + \overline{f(\bar{z})})$, $h(z) = \dfrac{1}{2i}(f(z) - \overline{f(\bar{z})})$.

Since $g(z)$, $h(z)$ are r-symmetric functions whose poles are contained in the set $\pi(f) \cup \pi(f)^*$, thus beyond X, (1) is valid on X too, so that obviously

$$\max\{\sup_{z \in X} |g(z)|, \sup_{z \in X} |h(z)|\} \leqslant \sup_{z \in X} |f(z)|.$$

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[1] We assume without loss of generality that all rational functions from this definition have the real coefficients.

Wherefrom it follows

$$\| g(A) \| \leqslant \sup_{z \in X} | g(z) | \leqslant \sup_{z \in X} | f(z) |, \quad \text{and}$$

$$\| h(A) \| \leqslant \sup_{z \in X} | f(z) |.$$

Finally $\| f(A) \| = \| g(A) + ih(A) \| \leqslant 2 \sup_{z \in X} | f(z) |$. □

Concerning this, a question arises: *Is it possible in a general case to decrease the constant* $\alpha = 2$ *from above proposition* $(1 \leqslant \alpha < 2)$? If the answer is affirmative and moreover $\alpha = 1$, q-spectral and c-spectral sets are the same:

Proposition 3. *A closed set* X *of the complex plane is spectral*:

a) *if and only if the set* X^* *is spectral*;

b) *if and only if* X *contains the spectrum* $\sigma(A)$ *and the set* $X \cup X^*$ *is spectral.*

Proof. We prove only the nontrivial part of the last statement.

Let $X \supseteq \sigma(A)$, $X \cup X^*$ be a spectral set, and $f(z)$ be any r-symmetric rational function whose poles lie beyond X.

Since the set $\pi(f)$ does not have a common element with $X \cup X^*$ we have

$$\| f(A) \| \leqslant \sup_{z \in X \cup X^*} | f(z) |.$$

But since

$$\sup_{z \in X \cup X^*} | f(z) | = \max \{ \sup_{\lambda \in X} | f(\lambda) |, \sup_{\lambda \in X^*} | f(\lambda) | \} = \sup_{z \in X} | f(z) |,$$

it follows $\| f(A) \| \leqslant \sup | f(z) |$, q. e. d. □

Corollary 1. *Let the closed set* X *be a spectral set of* A. *Then the sets* $X^+ = (X \cap C^+) \cup \sigma(A)$ *and* $X^- = (X \cap C^-) \cup \sigma(A)$ *are spectral sets too.*

Proof. Since the closed set X^+ contains the spectrum $\sigma(A)$, $X = X^+ \cup (X^+)^*$ and $X^- = (X^+)^*$, the statement follows from Proposition 3. □

In this way we may considerably reduce a fixed spectral set X, to get such a set again.

Concerning this, it can be of some interest the next problem: *Does there exist at least one spectral set* X *such that the set* $X \cap X^*$ *is not spectral*?

All examples which we know are negative. Nevertheless we propose that the answer is affirmative. If for instance, there exists an operator such that spectrum $\sigma(A)$ is not a spectral set, but $\sigma(A) \cup R$ is spectral — (with assumption $\sigma(A)$ and R do not have a common element), then the answer is affirmative.

The following three statements are standard.

Proposition 4. *The intersection of all spectral sets coincides with the spectrum* $\sigma(A)$.

If the operator A *is normal, a closed set* X *is spectral if and only if it is* c*-spectral.*

Proof. The first property is obvious, having in mind the corresponding property of c-spectral sets.

Further, since the spectrum $\sigma(A)$ of a normal operator is its c-spectral set, the second statement is evident too. □

Proposition 5. *The unit disc $|z| \leqslant 1$ is a spectral set of A if and only if A is a contraction operator.*

Proof. As in the Hilbert space, the condition $\|A\| \leqslant 1$ is necessary. Conversely, if A is a contraction, then the unit disc is a c-spectral, thus a q-spectral set also. □

Proposition 6. (The spectral mapping theorem). *Let $\omega(z)$ be any r-symmetric rational function having no poles on $\sigma(A)$, and X be any r-symmetric spectral set of A.*

Then $\omega(X)$ is such a set of the operator $\omega(A)$. □

Proposition 7. (a) *A disc $|z-z_0| \leqslant r$, whose center z_0 is on the real axis, is a spectral set of A if and only if $\|A-z_0 I\| \leqslant r$.*

(b) *Its exterior $|z-z_0| \geqslant r$ is a spectral set of A if and only if $\|(A-z_0 I)^{-1}\| \leqslant 1/r$.*

(c) *The unit circle $|z|=1$ is spectral set of A if and only if A is unitary.*

(d) *Imaginary axis $Jm = iR$ is a spectral set of A if and only if A is skew-symmetric.*

(e) *The real axis R is a spectral set of A, if A is symmetric.*

Proof. The proofs for (a), (b) (c), are standard.

If A is symmetric or skew-symmetric, then it is normal, so that its spectrum is a spectral set. Therefore $X=R$ (or $X=Jm$ respectively) is a spectral set of A.

Suppose $Im=iR$ is a spectral set of A.

Let us consider the function $\omega(z)=(z+1)(z-1)^{-1}$. It is r-symmetric and defined on the set $X=iR$. It is easily seen that it maps the set X on the set $\{|z|=1, z\neq 1\}$. By the spectral mapping theorem, this set, and therefore the unit circle $|z|=1$, is a spectral set of $\omega(A)=(A+I)(A-I)^{-1}$. But then $\omega(A)$ must be unitary, wherefrom by using the standard arguments, we conclude that $\mathrm{Re}\langle Ax, x\rangle = 0$ $(x \in H)$. Thus $\mathrm{Re}(A)=0$ so that A is skew-symmetric. □

We remark that the last statement (e) is not complete.

By analogy, it can be expected that *the real axis R is a spectral set of symmetric operators only.* But this fact still awaits its solution. Even, we think that *it is not true,* and that it must be a more large class of operators. But so far without complete solution.

Concerning that, we conclude with the next unsolved questions:

(1) When *the real line R* is a spectral set of an operator?

(2) When *a line* (*in a general position*) is a spectral set?

(3) When *a disc* (*in a general position*) is a spectral set?

In that direction, we have a partial result only.

Statement 1. *If R is a spectral set of an operator A, then*

$$(\mathrm{Re}\langle Ax, x\rangle)^2 \leqslant \mathrm{Re}\langle A^2 x, x\rangle \|x\|^2 \qquad (x\in H).$$

Consequently, the operator $\mathrm{Re}(A^2)$ *must be positive.*

Proof. Let us consider the function

$$f(z)=\frac{1}{(z+n)^2+m^2}=\frac{1}{(z+n+im)(z+n-im)},$$

where $z_0=n+im\in C\setminus R\ (m\neq 0)$.

Then $f(z)$ is r-symmetric, defined on $X=R$, and $\max_{z\in R}|f(z)|=1/m^2$. Consequently, for operators $f(A)=((A+nI)^2+m^2I)^{-1}$ we have $\|f(A)\|\leqslant 1/m^2$, i.e.

$$\|(A+nI)^2x+m^2x\|\geqslant m^2\|x\| \qquad (x\in H).$$

From that relation we obtain

$$\|(A+nI)^2x\|^2\geqslant -2m^2\,\mathrm{Re}\langle (A+nI)^2x, x\rangle \qquad (x\in H),$$

which is possible (for every real $m\neq 0$) only if $\mathrm{Re}\langle (A+nI)^2x, x\rangle\geqslant 0$. It follows that

$$n^2\|x\|^2+2n\,\mathrm{Re}\langle Ax, x\rangle+\mathrm{Re}\langle A^2x, x\rangle\geqslant 0,$$

which is possible (for every real n) only if

$$(\mathrm{Re}\langle Ax, x\rangle)^2-(\mathrm{Re}\langle A^2x, x\rangle)\|x\|^2\leqslant 0 \qquad (x\in H).$$

If we put $S=\mathrm{Re}(A)$, $T=Jm(A)$, the last relation gets the form

$$\|Tx\|^2\leqslant\|Sx\|^2-\langle Sx, x\rangle^2/\|x\|^2 \qquad (x\in H\setminus 0). \ \square$$

In [1] C. Foiaş has obtained a charasteristic property of complex Hilbert spaces. For Wachs spaces the analogue is valid.

Proposition 8. *Let H be a quaternionic Banach space. If the unit disc is a spectral set of every contraction on H, then H is a Wachs space.*

Proof. The proof is a modification of Foiaş's proof, and the consequence of the following fact: The parallelogram identity in H^s, thus in H, is a sufficient condition for H to be a Wachs space. $\square$

In a finite-dimensional complex Hilbert space the next is true: spectrum $\sigma(A)$ is a spectral set if and only if A is normal (von Neumann [3], or [4], p. 434). In our situation the same is valid.

Proposition 9. *Let H be a finite-dimensional Wachs space. Then $\sigma(A)$ is a spectral set of A if and only if A is normal.*

Proof. Suppose $\sigma(A)$ is a spectral set of A. If H is a n-dimensional space, spectrum $\sigma(A)$ consists of the exactly $2n$ proper values:

$$X=\sigma(A)=\{\xi_1,\dots,\xi_n;\ \bar\xi_1,\dots,\bar\xi_n\} \qquad (\xi_j\in C^+).$$

Let $z_1, z_2, \dots, z_{n_0}$ be all different proper values of A. Then for every $k=1,\dots,n_0$ there exists a polynomial $p_k(z)$ of degree n_0-1 such that

$$p_k(z)=\begin{cases} z_k, & \text{if } z=z_k \\ \bar{z}_k, & \text{if } z=\bar{z}_k=z_{k'} \\ 0, & \text{if } z=z_j\neq z_k,\ \bar{z}_k \end{cases}.$$

That polynomial is unique, and by using the Cramer's rule it can be shown that all their coefficients are real. Thus all $p_k(z)$ are r-symmetric.

By the spectral mapping theorem, we have that $p_k(X)=\{0, z_k, \bar{z}_k\}$ is a spectral set of $B_k=p_k(A)$.

For proving that all operators B_k $(k=1,\dots,n_0)$ are normal, it is suitable to differ the following cases.

1. $z_k=0$. Then the set $\{-1\}$ is a spectral set of $\omega(B_k)=B_k-I$, so that $\omega(B)$ must be unitary. Therefore it follows that B_k is normal, thus symmetric.

2. $z_k\in R\backslash 0$. In this case the set $\{-1, 1\}$ is a spectral set of $\omega(B_k)=(2/z_k)B_k-I$, so that $\omega(B_k)$ is unitary. It implies that $B_k=(z_k/2)(I+U_k)$ is normal. Moreover, since the spectrum of $\omega(B_k)$ is real, B_k is symmetric.

3. $z_k\in Jm\backslash 0$. Then the imaginary axis $Jm=iR$ is a spectral set of B_k, thus (by Proposition 7) B_k is skew-symmetric.

4. $z_k\in C\backslash(R\cup Jm)$. If $z_k=x_k+iy_k$, let us consider the function $\omega(z)=t_0^{-1}z-1$, where $t_0=(x_k^2+y_k^2)/2x_k$ $(x_k\neq 0)$. It maps the spectral set $\{0, z_k, \bar{z}_k\}$ of B_k in a spectral set $\{-1, \varepsilon_k, \bar{\varepsilon}_k\}$ $(|\varepsilon_k|=1)$ of $\omega(B_k)=t_0^{-1}B_k-I$. Concequently, $\omega(B_k)$ must be unitary, thus B_k is normal.

In such a manner, we see that all operators B_k $(k=1,\dots,n_0)$ are normal.

Put: $q(z)=p_1(z)+\dots+p_{n_0}(z)$. It follows immediately that $q(z_k)=z_k$ $(k=1,\dots,n_0)$, thus $q(z)\equiv z$ (because $\deg(q)=n_0-1$). Therefore we obtain $A=\sum_{k=1}^{n_0} p_k(A)$. Since $A^*=\sum_{k=1}^{n_0} p_k(A)^*$ we find:

$$A^*A=\sum_{k=1}^{n_0} p_k(A)^*p_k(A)+\sum_{k\neq l}(p_k(A)^*p_l(A)+p_l(A)^*p_k(A)), \tag{1}$$

$$AA^*=\sum_{k=1}^{n_0} p_k(A)p_k(A)^*+\sum_{k\neq l}(p_k(A)p_l(A)^*+p_l(A)p_k(A)^*). \tag{2}$$

Since $p_k(A)$ and $p_l(A)$ are commutative and normal, and the adjoint operator P^* of some $P\in B(H)$ coincides with the adjoint of P in H^s, we can apply a theorem of Fugled (C. R. Putnam — "Commutation properties of Hilbert space operators", Springer-Verlag, Berlin, 1967; Theorem 1.6.1., p. 9).

We obtain that

$$p_k(A)^*p_l(A)+p_l(A)^*p_k(A)=p_l(A)p_k(A)^*+p_k(A)p_l(A)^*,$$

so that A is a normal operator, q. e. d. □

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