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ON A PROPERTY OF PARTIALLY ORDERED SETS

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§ 0. Introduction

In what follows we shall deal with a problem which professor Đ. Kurepa formulated in his seminar „The Set Theory" which was held in 1976/1977 and also in his paper [5] (problem (6 : 4 : 2)):

Problem 0.1. Does every partially ordered set have the property (K)?

(Definition of the condition (K) is given in § 1). This problem naturally arises at the analysis of natural linear extension of pseudotree (see [5], §§ 5, 6). At this natural linear extension of pseudotree some of its particular subsets, so called strictly convex sets (definition 1.2) are transformed into convex subsets of this linear extension (Theorem 1.3). So, the subject of the problem above is the question whether every partially ordered set has linear extension in which the strict convexity will be preserved.

§ 1. Pseudointervals and strictly convex sets

We conceive of partially ordered set as an ordered pair $P=(E, <)$ where $<\subseteq E^2$ is antireflexive, antisymmetric and transistive relation. Partially ordered sets we shall denote by $L, P, Q, \ldots, L_1, P_1, Q_1, \ldots$, and corresponding relations by $<_L, <_P, <_Q, \ldots, <_{L_1}, <_{P_1}, <_{Q_1}, \ldots$. In the cases, where it is possible, instead of $<_{P\alpha}$ we can write simply as $<_\alpha$. For $a, b\in E$ and $a\neq b$ let us define relation $\|$ like: $a\|b$ if and only if $(a, b)\notin <$ and $(b, a)\notin <$. Let $a\in E$; then we put $[a]_P := \{x \therefore x\in E$ and $x<a \vee x>a \vee x=a\}$. For partially ordered set $Q=(E, <Q)$ we shall say that it is an extension of partially ordered set $P=(E, <_P)$ if $<_P\subseteq <_Q$. Particulary we shall consider linear extensions, that is the case when Q is linear ordered set.

Now, let us write down main definitions of this section taken from [5], § 6.

Definition 1.1. Pseudointerval of partially ordered set $P=(E, <)$ with ends $a, b, a\leqslant b$ denoted by $]a, b[$ is the set

$$]a, b[:= \{x \therefore x\in E,\ (a\leqslant x)\wedge\neg(x>b) \text{ or } (x\leqslant b)\wedge\neg(x<a)\}$$

or shortly

$$]a, b[:= ([a]_P\cup[b]_P)\setminus((\cdot, a)_P\cup(b, \cdot)_P).$$

D e f i n i t i o n 1.2. Let $P=(E; <)$ be a partially ordered set, subset J is called stiictly convex set in P if from $a, b \in J$, $a \leqslant b$ follows that $]a, b[\subseteq J$ and J cannot be decompose dinto two disjoint parts $J_1 \neq \varnothing \neq J_2$ such that every point of J_1 is incomparable with every point in J_2.

We can verify that for linear ordered sets pseudointervals coincides with closed intervals and strictly convex subsets coincides with convex subsets and also, in general case, pseudointerval need not be a strictly convex set.

Pseudotree or ramified set is defined (see [2] [3] [4]) like partially ordered set $(R, <)$ such that for every $x \in R$ the set $(\cdot, x]_{(R, <)}$ is a chain. If that set is furthermore well-ordered then pseudotree $(R, <)$ we call tree (see [5] and also [3], § 8. A). We can be easily convinced that, if $(R, <)$ is pseudotree then every set $J \subseteq R$ written in the form $J=\bigcup_{x \in L}[x, \cdot)_{(R, <)}$ where L is a chain, is a strictly convex subset. Therefore, especially, the sets formed by $[x, \cdot)_{(R, <)}$ are strictly convex for every $x \in R$.

Let $P=(E, <)$ be an arbitrary partially ordered set. Collection of all strictly convex subsets in P we shall denote by $\mathcal{K}(P)$. Let us observe that for every $x \in E$ $\{x\} \in \mathcal{K}(P)$ holds.

For partially ordered set $P=(E, <_P)$ let us observe the following properties which we shall meet later:

(K) There exists a linear extension $L=(E, <_L)$ of partially ordered set P such that $\mathcal{K}(P) \subseteq \mathcal{K}(L)$ hold.

(S) For every three points $a, b, c \in E$, such that $a <_P b$, $a \|_P c$ and $b \|_P c$, there exists a strictly convex set $J \subseteq E$ such that $a, b \in J$ but $c \notin J$.

Partially ordered sets with properties (K), (S) we shall call (K)-, (S)-partially ordered sets iespectively. Every pseudotree $(R, <)$ is (S)-partially ordered set. We shall say that for $a, b, c \in R$, $a<b$, $a \| b$ and $b \| c$ desired strictly convex set is $[a, \cdot)_{(R, <)}$.

That for every pseudotree $(R, <)$ the answer problem 0.1. is positive states the following theorem.

T h e o r e m 1.3. *If $L=(R, <_L)$ is an arbitrary natural linear extension of any pseudotree $(R, <)$, then $\mathcal{K}(R, <) \subseteq \mathcal{K}(L)$.*

P r o o f: Let us assume the contrary – that there exists $J \in \mathcal{K}(R, <) \setminus \mathcal{K}(L)$. So, there exist three different points $a, b, c \in R$ such that $a <_L c <_L b$, $a, b \in J$ but $c \notin J$. Let us put

$$J_1:=\{x \in J \therefore x <_L c\} \text{ and } J_2:=\{x \in J \therefore c <_L x\}.$$

So, $J_1 \cup J_2=J$, $J_1 \cap J_2=\varnothing$ and since $a \in J_1$ and $b \in J_2$ it follows that J_1 and J_2 are non empty. According to definition 1.2. we conclude that there exist points $a_1 \in J_1$ and $b_1 \in J_2$ such that $a_1<b_1$. Since $c \notin J$ and $]a_1, b_1[\subseteq J$, we have that $c \notin]a_1, b_1[$ which, considering that $a_1 <_L c <_L b$ and the fact that L is an extension of $(R, <)$, means that $a_1 \| c$ and $b_2 \| c$. So especially we have that $c \notin [a_1, \cdot)_{(R, <)}$. Since L is natural linear extension of $(R, <)$ using the theorem (5 : 1) II from [5], it follows that $[a_1, \cdot)_{(R, <)}$ is convex in L, in contradiction with $c \notin [a_1, \cdot)_{(R, <)}$, $b_1 \in [a_1, \cdot)_{(R, <)}$ and $a_1 <_L c <_L b$. The contradiction proves the theorem.

§ 2. Basic results

We shall write down some definitions taken from [1].

Let $\mathcal{P}=\{P_\alpha=(E, <\alpha) \therefore \alpha\in A\}$ be an arbitrary collection of partially ordered sets on the same set E. The product of collection $\mathcal{P}$ is the partially ordered set defined by

$$\prod_{\alpha\in A} P_\alpha := (E, \bigcap_{\alpha\in A} <_\alpha).$$

Especially, for $\mathcal{P}=\{P_1, P_2\}$ we put $P_1 P_2 := (E, <_1 \cap <_2)$.

By $\dim P$, where $P=(E, <)$ is an arbitrary partially ordered set, we denote cardinal function defined by $\dim P := \min\{|\mathcal{L}| \therefore \mathcal{L}$ is a collection of linear extensions of P and $P=\prod_{L\in\mathcal{L}} L\}$.

Let $P=(E, <_P)$ and $Q=(E, <_Q)$ be two partially ordered sets on a same set E. We shall say that P and Q are reciprocally conjugate if two arbitrary different points $a, b\in E$ are ordered in only one of these partially ordered sets, in other words holds: $a<_P b$ or $b<_P a$ if and only if $a\|_Q b$. It is clear that in the last equivalance P and Q can exchange their places. Partially ordered set is called reversible if it has a conjugate.

Lemma 2.1. ([1], *Lemma* 35.1). *If partially ordered set P and are conjugate then* $L_1=(E, <_P\cup <_Q)$ *and* $L_2=(E, <_P \cup (<_Q)^{-1})$ *are linear extension of P and* $P=L_1 L_2$.

Lemma 2.2 ([1], *Theorem* 36.1, (1) $\Leftrightarrow$ (3)). *Let* $P=(E, <)$ *be an arbitrary partially ordered set, then* $\dim P\leqslant 2$ *if and only if P is reversible.*

Proof: One side of equivalence follows according to lemma 2.1. so let us prove another one. Let us assume that $\dim P\leqslant 2$ which means that there exist linear extensions $L_1=(E, <_1)$ and $L_2=(E, <_2)$ of P such that $P=L_1 L_2$. Let us determine the partially ordered set Q by $Q=(E, <_1\cap(<_2)^{-1})$ and prove that Q is conjugate with P. If a and b are arbitrary two different points of E and let, for determination, be $a<_1 b$, then it holds either $a<_2 b$, that is $a<_P b$ or $b<_2 a$, that is $a<_Q b$ which means that the pair (a, b) is in only one of the relations $<_P$ and $<_Q$. So, arbitrary two points a and b in E are ordered in one and only one of the relations $<_P$ and $<_Q$. Q.E.D.

An answer to the problem 0.1. is given by following theorem:

Theorem 2.3. *Every partially ordered set* $P=(E, <_P)$, *such that* $\dim P\leqslant 2$, *satisfies condition* (K).

Proof: Let $\dim P\leqslant 2$. According to lemma 2.2. there exists a partially ordered set $Q=(E, <_Q)$ which is conjugate with P. According to lemma 2.1. we conclude that $L=(E, <_P\cup <_Q)$ is a linear extension of P and Q. Let us prove that P and L satisfy condition (K) (see § 1.). Let us assume the contrary, that is, L along with P does not satisfy condition (K), which means that there exists strictly convex set $J \subseteq E$ of P which is not strictly convex in L. Hence, there exist three reciprocally different points $a, b, c \in E$ such that holds $a <_L c <_L b$, $a, b \in J$ and $c \notin J$. Let us put

$$J_1 = \{x \in J \therefore x <_L c\}, \quad J_2 = \{x \in J \therefore c <_L x\}$$

then $J = J_1 \cup J_2$, $J_1 \varnothing \neq J_2$ and $J_1 \cap J_2 = \varnothing$. By definition 1.2 of strictly convex set we conclude that there exist $a_1 \in J_1$ and $b_1 \in J_2$, such that $a_1 <_P b_1$, so we can put $a_1 = a$ and $b_1 = b$, that is to assume that $a <_P b$. We have that $]a, b[\subseteq J$ and therefore $c \notin]a, b[$. Since $a <_L c <_L b$ it must hold $a \|_P c$ and $b \|_P c$ and therefore $a <_Q c$ and $c <_Q b$, because of the fact that Q is conjugate with P and L is an extension of Q too. Because of transitivity of Q we have $a <_Q b$ which is contradictory with $a <_P b$. This contradiction proves the theorem.

Main Theorem 2.4. *Let $P = (E, <_P)$ be an arbitrary (S)-partially ordered set, then P satisfies condition (K) if and only if* $\dim P \leqslant 2$.

Proof: According to theorem 2.3. it remains to be shown that every partially ordered set P which satisfies condition (S) and (K) has dimension $\leqslant 2$. Let $L = (E, <_L)$ be a linear extension of P, such that every strictly convex set in P is also convex in L. It is sufficient to prove that $Q = (E, <_L \setminus <_P)$ is partially ordered set and conjugate with P. Antireflexivity and antisymmetry of relation $<_Q = <_2 \setminus <_P$ is obvious, se let us only prove its transitivity. Let a, b, c be three different points of E such that $a <_Q c$ and $c <_Q b$ holds, then by definition of this relation must be $a <_L c$, $c <_L b$, $a \|_P c$ and $b \|_P c$, owing to the fact that L is linear extension of P. So, $a <_L b$ and in case that $a \|_P b$ we have that $a <_Q b$ what we needed. Let us consider the case when $a <_P b$. So, we have that for three different points a, b, c of E $a <_P b$, $a \|_P c$ and $b \|_P c$ hold and since P satisfies condition (S) there exists strictly convex set $J \subseteq E$ such that $a, b \in J$ and $c \notin J$ According to the condition (K) we have that J is also a convex set in L and, therefore, $[a, b]_L \subseteq J$. Since $c \in [a, b]_L \subseteq J$ we obtain contradiction with $c \notin J$. This contradiction proves the transitivity of relation $<_Q$.

Now, let us prove that P and Q are conjugate. Let $a, b \in E$ be two different points and, for determination, be $a <_L b$. In case that a and b are comparable in P, which means that $a <_P b$, we have, by definition of Q, that $a \|_Q b$. If a and b are incomparable in P, which means that $a \|_P b$, then by definition of Q must be $a <_Q b$. The proof is finished.

Consequence 2.5. Every pseudotree $(R, <)$ has dimension $\leqslant 2$.

Proof: in § 1. we have already proved that every pseudotree is (S)-partially ordered set and since according to theorem 1.3. we have get that it is also (K)-partially ordered set, the conclusions of the consequence 2.5 follows directly from the previous theorem 2.4.

Let $P = (E, <)$ be an arbitrary partially ordered set. We shall consider the following partially ordered set $(\mathcal{K}(P), \subset)$ where $\mathcal{K}(P)$ is the collection of all non empty strictly convex subsets in P defined in § 1. The following theorem gives us one more information about (K)-partially ordered sets.

Theorem 2.6. *Let $P=(E, <_P)$ be an arbitrary partially ordered set with property (K), then* $\dim(\mathcal{K}(P), \subset)\leqslant 2$.

Proof: Let $L=(E, <_L)$ be a linear extension of the set P, such that $\mathcal{K}(P)\subseteq\mathcal{K}(L)$. Let $L_1=(E\times\{0, 1\}, <_1)$ be lexicographical product of sets $L=(E, <_L)$ and $(\{0, 1\}, \{(0, 1)\})$ (i.e. $\{0, 1\}, \prec)$, where $\prec=\{(0, 1)\}$). For every $J\in\mathcal{K}(P)$ by J' where $J'\subseteq E\times\{0, 1\}=E'$ we denote the set which is obtained from $J\times\{0\}$ by convex closure, which means that

$$J'=\bigcup\{[a, b]_{L_1} \therefore a, b\in J\times\{0\}\}.$$

Let $\mathcal{K}'(P):=\{J' \therefore J\in\mathcal{K}(P)\}\,(\subseteq\mathcal{K}((L_1))$, then we directly verify that the partially ordered sets $(\mathcal{K}(P), \subset)$ and $(\mathcal{K}'(P), \subset)$ are isomorphic. Let $L_2=(E'\times\{0, 1\}, <_2)$ be the lexicographical product of linear ordered sets $L_1=(E' <_2)$ and $(\{0, 1\}, \{(0, 1)\})$. Similarly as for $J'\in\mathcal{K}'(P)$ we define

$$J''=\bigcup\{[a, b]_{L_2} \therefore a, b\in J'\times\{1\}\}.$$

Let us put $\mathcal{K}'':=\{J'' \therefore J\in\mathcal{K}(P)\}\,(\subseteq\mathcal{K}(L_2))$. It is clear that the partially ordered sets $(\mathcal{K}'(P), \subset)$ and $(\mathcal{K}''(P), \subset)$ are reciprocally isomorphic which according to the above means that $(\mathcal{K}(P), \subset)$ and $(\mathcal{K}''(P), \subset)$ are isomorphic. According to the above construction we verify that the collection $\mathcal{K}''(P)$ satisfies the following condition: (∗). If a point $a''\in J_1''\setminus J_2''$ is to the right (left) of convex set J_2'', where $J_1'', J_2''\in\mathcal{K}''(P)$, then there exists a point $b''\in J_2''\setminus J_2''$ which is different from a'' and is also to the right (left) of J_2''.

Let $a''\in J_1''\setminus J_2''$ be to the right of J_2'', which by definition means that there exists points $c', d'\in J_1'$, such that $a''\in[c'\times\{1\}, d'\times\{1\}]_{L_2}$, So, it must be that $c'\in J_1'\setminus J_2'$ and c' is to the right of J_2''. Applying again the above definition we conclude that there exist $e, f\in J_1$ such that $c'\in[e\times\{0\}, f\times\{0\}]_{L_1}$. So, necessarily $e\in J_1\setminus J_2$ and e is to the right of J_2''. By definition of lexicographical order and definition of the sets J' and J'' we conclude that $(e\times\{0\})\times\{1\}$ and $(e\times\{1\})\times\{1\}$ are two different points in $J_1''\setminus J_2''$ being to the right of J_2''. For desired point b'' we can take one which is different from a''. The case when $a''\in J_1''\setminus J_2''$ is to the left of J_2'' can be settled analogously.

Let $\tilde{L}_2=(\tilde{E}'', <)$ be completification of linear ordered set $L_2=(E'', <_2)$ in the sense of Dedekind, where we put $E''=E'\times\{0, 1\}$. Let us define the collection $\tilde{\mathcal{K}}(P)$ by

$$\tilde{\mathcal{K}}(P):=\{\bar{J}'' \therefore J''\in\mathcal{K}''(P)\},$$

where $\bar{J}''$ denotes the closure in order topology of linear ordered set $\tilde{L}_2$. Using the property (∗) of collection $\mathcal{K}''(P)$ we directly check, that $(\mathcal{K}''(P), \subset)$ is a partially ordered set isomorphic to $(\tilde{\mathcal{K}}(P), \subset)$. Since $\tilde{L}_2$ is complete linear ordered set we conclude that the collection $\tilde{\mathcal{K}}(P)$ consists of closed intervals in $\tilde{L}_2$, that is, every $\tilde{J}\in\tilde{\mathcal{K}}(P)$ is of a form $[x, y]_{\tilde{L}_2}$. Finally we have that $(\mathcal{K}(P), \subset)$ is isomorphic to $(\tilde{\mathcal{K}}(P), \subset)$ namely to some collection of closed intervals of a linear ordered set $\tilde{L}_2$ which according to [1] (Theorem 36.1, (3) ⇔ (4)) means that $\dim(\mathcal{K}(P), \subset)\leqslant 2$, that was to be proved.

Using the previous theorem we can prove the consequence 2.5, that the dimension of an arbitrary pseudotree is $\leqslant 2$, on a different way. It is known that $(R, >)$ is isomorphic to $(\{[x, \cdot)_{(R,<)} \therefore x \in R\}, \subset)$ and as we indicated in § 1. that

$$\{[x, \cdot)_{(R,<)} \therefore x \in R\} \subseteq \mathcal{K}(R, <)$$

and since, according to the theorem 1.3., $(R, <)$ is a (K)-partially ordered set, we have, by previous theorem, that

$$\dim (R, <) = \dim (\{[x, \cdot)_{(R,<)} \therefore x \in R\}, \subset) \leqslant \dim (\mathcal{K}(R, <), \subset) \leqslant 2,$$

what we needed.

But, the converse holds as well, i.e., if $\{[x, \cdot)_{(R,<)} \therefore x \in R\} \subseteq \mathcal{K}(R, <)$, then $(R, <)$ is a pseudotree.

In connection with the problem 0.1. naturally appears the following problem:

Problem 2.7. Does for every partially ordered set $P = (E, <)$, $\dim \mathcal{K}(P), \subset) \leqslant 2$ hold?

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