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A GENERALIZATION OF BANACH'S CONTRACTION PRINCIPLE

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Abstract. In this paper we describe a class of conditions sufficient for the existence of a fixed point which generalize several known results. We introduced the concept of a f-contraction and generalized f-contraction T of a metric space X into itself. Banach's contraction principle can be extended to (generalized) f-contractions. In other words, a fixed point theorem for relaxed generalized f-contractions is proved, and an example is given to show the established results are indeed extensions.

1. Introduction and results

A number of authors have defined contractive type mappings on a complete metric space X which are gereralizations of the well-known Banach contraction, and which have the property that each such mapping has a unique fixed point. The fixed point can always be found by using Picard iteration, beginning with some initial choice $x_0 \in X$. The well-known Banach contraction principle is the following:

Let $T: X \to X$ be a mapping of a complete metric space (X, ρ) into itself. If T is a contraction, i.e. if

(A) $$\rho[Tx, Ty] \leqq \alpha \rho[x, y] \text{ for some } \alpha \in [0, 1),$$

and all $x, y \in X$, then:

(a) *T has a unique fixed point ξ in X;*

(b) *$T^n(x) \to \xi$ for all $x \in X$, and*

(c) *There exists an open neighborhood U of ξ such that for any neighborhood V of $T\xi$ there is an $n(V)$ which satisfies $n \geqq n(V) \Rightarrow T^n(U) \subset V$, or*

$$T^n x \in K(\xi, \alpha^n (1-\alpha)^{-1} \rho[x, Tx]),$$

for every $x \in X$, and $n \in N$, where K is closed ball.

In other words, if T is a contraction mapping on a complete metric space X, then the equation $Tx=x$ has in X a unique solution. The theorem of Banach and its extensions usualy are proved by the fact that the geometrical series is convergent. Some different proof of the Banach theorem is given by R. Kannan, where the investigated properties of subsets of X, defined as $S_\gamma \overset{\text{def}}{=} \{x \in X : \rho[x, Tx] \leqq \gamma\}$, $\gamma > 0$. For extension of Banach contraction principle and certain other related results, see references.

In [25] we have proved the following theorem:

Theorem T. *Let $T: X \to X$ be a mapping on X and let X be a T-orbitally complete metric space. If T satisfies the following condition: there exist real numbers α_i, β for every $x, y \in X$ such that: $\alpha_1 + \alpha_2 + \alpha_3 > \beta$ and $\beta - \alpha_2 \geqq 0 \vee \beta - \alpha_3 \geqq 0$, and*

$$(T)\ \alpha_1 \rho[Tx, Ty] + \alpha_2 \rho[x, Tx] + \alpha_3 \rho[y, Ty] + \alpha_4 \min\{\rho[x, Ty], \rho[y, Tx]\} \leqq \beta \rho[x, y],$$

then for each $x \in X$, the sequence $(T^n x)$ converges to a fixed point of T.

Special cases of Theorem T have been discussed by Ivanov [14], Ćirić [10], R. Kannan [15], S. Reich [20], Bianchini [7], Rhoades [21], Hardy-Rogers [12], Kurepa [17], Rakotch [19], Boyd and Wong [5], I. Rus [22], and others (see references).

The essential in all so far published results in this domain is based, as it seems to us, on inequalites of the form $x_{n+1} \leqq \alpha x_n$ ($x_n \in \mathbf{R}_+ \overset{\text{def}}{=} (0, +\infty)$, $\alpha \in [0, 1)$) and on Picard's method. Having this fact in view, we started from a result of S. Prešić [17]. Our first step was a theorem, proved in the papers [24], [25], concerning ordered sets and generalized difference relations; from it we have deduced the following result:

Proposition 1. ([25], p. 236.) *Let $f: \mathbf{R}_+^{k+1} \to \mathbf{R}_+$ ($k \in N$) be monotonically increasing (with respect to every real argument) and semihomogeneous mapping, and let the sequence (x_n) of nonnegative real numbers satisfy the condition*

$$(1) \qquad x_{n+k} \leqq f(\alpha_0 x_n, \alpha_1 x_{n+1}, \ldots, \alpha_k x_{n+k}), \quad n \in N;$$

k fixed natural number, where $\alpha_0, \alpha_1, \ldots, \alpha_k$ are nonnegative real constants and $f(\alpha_0, \alpha_1, \ldots, \alpha_k) \in (0, 1)$. Then, there exists numbers $\mathscr{L} > 0$ and $\theta \in (0, 1)$ such that

$$(2) \qquad x_n \leqq \mathscr{L} \theta^n \ (n = 1, 2, \ldots), \ \mathscr{L} = \max_{i=1,\ldots,k} \{x_i \theta^{-i}\}.$$

The proof of this proposition is given in [25] and otherwise it is perfoımed by the application of mathematical induction.

We say that the mapping $f: \mathbf{R}_+^k \to \mathbf{R}_+$ ($k \in N$) is semihomogeneous iff

$$f(\delta x_1, \ldots, \delta x_k) \leqq \delta f(x_1, \ldots, x_k), \ \delta > 0.$$

It is clear that the condition of semihomogeneity implies homogeneity for the mapping f.

1.1. *In this paper we introduced the concept of a f-contraction T of a metric space X into itself i. e. of a mapping $T: X \to X$ such that for every $x, y \in X$ there exists nonnegative real numbers $\alpha_i(x, y) = \alpha_i$ $(i = 1, \dots, 5)$ such that*

$$\text{(B)}\quad \rho[Tx, Ty] \leqq f(\alpha_1 \rho[x, y],\ \alpha_2 \rho[x, Tx],\ \alpha_3 \rho[y, Ty],\ \alpha_4 \rho[x, Ty],\ \alpha_5 \rho[y, Tx]),$$

where $\sup\{f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) : x, y \in X\} = \lambda \in [0, 1)$ *and the existing mapping* $f: (\mathbf{R}_+^0)^5 \to \mathbf{R}_+^0 \overset{\text{def}}{=} [0, +\infty)$ *is increasing and semihomogeneous.*

Let T be a mapping of a metric space X into itself. For $A \subset X$, let

$$\sigma(A) = \sup\{\rho[a, b] : a, b \in A\}$$

and for each $x \in X$, let

$$O(T^m x,\ n) = \{T^m x, T^{m+1} x, \dots, T^{m+n} x\},\ m = 0, 1, \dots;\quad n \in N,$$

$$O(T^m x,\ \infty) = \{T^m x, T^{m+1} x, \dots\},\ m = 0, 1, \dots,$$

where it is understood that $T^0 x = x$. A space X is said to be T-orbitally complete iff every Cauchy sequence which is contained in $O(x, \infty)$ for some $x \in X$ converges in X (c. f. [25]).

Further, as a corollary of the last result, we got the following statement.

Lemma 1. (*Corollary of the Proposition* 1, see [25]).

Let $T: X \to X$ be a f-contraction on X and let n be any positive integer. Then for each $x \in X$ and all positive integers i and j

(a) $$1 \leqq i,\ j \leqq n \Rightarrow \rho[T^i x, T^j x] \leqq \lambda \sigma[O(x, n)];$$

(b) $$(\forall x \in X)\ (\exists k \leqq n)\ \rho[x, T^k x] = \sigma[O(x, n)];$$

(c) $$\sigma[O(x, \infty)] \leqq (1 - \lambda)^{-1} \rho[x, Tx].$$

Now, we can formulate a corresponding statement for f-contractive mappings.

Theorem 1. *Let T be a f-contraction on a metric space X and let X be T-orbitally complete. Then*:

(a) *T has a unique fixed point ξ in X*;

(b) $\lim T^n x \to \xi$ *for all $x \in X$, and*

(c) $T^n x \in K(\xi, \lambda^n (1 - \lambda)^{-1} \rho[x, Tx])$ *for every $x \in X$,*

and $n \in N$, where K is closed ball.

But, if we wish to weaken the condition of semihomogenity, replacing semihomogenity on $\mathbf{R}_+ \overset{\text{def}}{=} (0, \infty)$ by semihomogenity on $[\alpha, +\infty) \subset \mathbf{R}_+$ we obtain the definition semihomogenity of order $\alpha \geqq 1$. Precisely said, the mapping $f: \mathbf{R}_+^k \to \mathbf{R}_+$ $(k \in N)$ is semihomogeneous of order $\alpha \geqq 1$ iff $f(\delta x_1, \ldots, \delta x_k) \leqq \leqq \delta f(x_1, \ldots, x_k)$, $x_1 = \cdots = x_k = 1$, for each $\delta \in [\alpha, +\infty)$ and any $\alpha \geqq 1$. Then occurs the deciding moment when a sequence (x_n) defined as in (1) loses the property of tending to zero with a geometric velocity, but still converges to zero. In connexion with that, we shall prove the following.

Proposition 2. *Let the mapping* $f: \mathbf{R}_+^{k+1} \to \mathbf{R}_+$ $(k \in N)$ *be increasing in each coordinate variable, semihomogeneous of order* $\alpha \geqq 1$, *and with the properties* $(\forall t \in (0, \alpha])\ (f(t, \ldots, t) < t \wedge \limsup_{y \to t+0} f(y, \ldots, y) < t)$, *and let the sequence* (x_n) *of nonnegative real numbers satisfy the condition*

$$x_{n+k} \leqq f(x_n, x_{n+1}, \ldots, x_{n+k}), \quad n \in N, \tag{3}$$

k being a fixed natural number. Then the sequence (x_n) *tends to zero.* The velocity of this convergence is not necessarily geometrical. The proposition holds even when $\alpha = +\infty$.

In connexion with this, we shall introduce the concept *of generalized f-contraction T of a metric space X into itself i. e. of a mapping* $T: X \to X$ *such that for all* $x, y \in X$,

$$\rho[Tx, Ty] \leqq f(\rho[x, y], \rho[x, Tx], \rho[y, Ty], \rho[x, Ty], \rho[y, Tx]), \tag{C}$$

where the existing mapping $f: (\mathbf{R}_+^0)^5 \to \mathbf{R}_+^0 \overset{\text{def}}{=} [0, +\infty)$ *is increasing, semihomogeneous of order* $\alpha \geqq 1$, *and with the properties*

$$(\forall t \in (0, \alpha])(f(t, \ldots, t) < t \wedge \limsup_{y \to t+0} f(y, \ldots, y) < t).$$

We assume that the case $\alpha = +\infty$ is also possible.

And finally, at the next step we prove a very general fixed point theorem which generalizes great numbers of known results.

Theorem 2. *Let T be a generalized f-contraction on a metric space X and let X be T-orbitally complete. Then for each* $x \in X$, *the sequence* $(T^n x)$ *converges to a fixed point of T.*

In [26] we have proved the following theorem.

Theorem. *T* 1. *Let T be a mapping of a metric space X into itself and let X be T-orbitally complete. Suppose that there exists a self map f on* $\mathbf{R}_+^0$ *such that f is* $(\forall t \in R_+) f(t) < t$, $\limsup_{y \to t+0} f(y) < t\ (t \in R_+)$ *and with the property*

$$\rho[Tx, Ty] \leqq f(\Delta),\ \Delta \in \{\rho[x, y], \rho[x, Tx], \rho[y, Ty], \rho[x, Ty], \rho[y, Tx]\}, \tag{D}$$

for each $x, y \in X$. *Then for each* $x \in X$, *the sequence* $(T^n x)$ *converges to a fixed point of T.*

The proof of this theorem is given in [26] and it is based upon the proposition 3, proved in [26].

P r o p o s i t i o n 3. *Let the mapping* $f\colon R_+ \to R_+ \overset{\text{def}}{=} (0, +\infty)$ *with the properties* $(\forall\, t \in R_+)\, f(t) < t$ *and* $\limsup\limits_{y \to t+0} f(y) < t\ (t \in R_+)$, *and let the sequence* (x_n) *of nonnegative real numbers satisfy the condition* $x_{n+1} \leqq f(x_n)$, $n \in N$; *then the sequence* (x_n) *tends to zero.*

Special cases of f-contraction or generalized f-contraction have been discussed by

(4) (Rakotch [19]) There exists a monotone decreasing function $\alpha\colon (0, \infty) \to [0, 1)$ such that, for each $x, y \in X$, $x \neq y$,

$$\rho[Tx, Ty] \leqq \alpha\, \rho[x, y].$$

(5) (Edelstein [11]) For each $x, y \in X$, $x \neq y$,

$$\rho[Tx, Ty] < \rho[x, y].$$

(6) (Boyd and Wong [5]) There exists a continuous function φ on nonnegative reals numbers $\mathbf{R}_+$ satisfying $\varphi(t) < t$ for $t > 0$ such that for all $x, y \in X$

$$\rho[Tx, Ty] \leqq \varphi(\rho[x, y]).$$

(7) (Kannan [15]) There exists a number $\alpha \in (0, 2^{-1})$, such that, for each $x, y \in X$,

$$\rho[Tx, Ty] \leqq \alpha(\rho[x, Tx] + \rho[y, Ty]).$$

(8) (Bianchini [7]) There exists a number $\alpha \in [0, 1)$, such that for each $x, y \in X$,

$$\rho[Tx, Ty] \leqq \alpha \max\{\rho[x, Tx], \rho[y, Ty]\}.$$

(9) (Reich [20], Rus [22]) There exists nonnegative numbers a, b, c satisfying $a + b + c < 1$ such that, for each $x, y \in X$,

$$\rho[Tx, Ty] \leqq a\,\rho[x, Tx] + b\,\rho[y, Ty] + c\,\rho[x, y].$$

(10) (Sehgal [23]) For each $x, y \in X$, $x \neq y$,

$$\rho[Tx, Ty] < \max\{\rho[x, y], \rho[x, Tx], \rho[y, Ty]\}.$$

(11) (Rhoades [21], Chatterjea [8]) There exists a number $h \in [0, 1)$ such that, for each $x, y \in X$,

$$\rho[Tx, Ty] \leqq h \max\{\rho[x, Ty], \rho[y, Tx]\}.$$

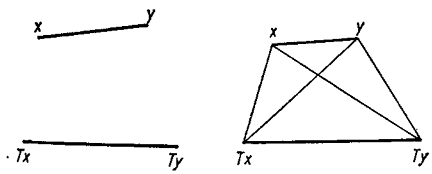
(12) (Hardy and Rogers]12]) There exist nonnegative constants a_i satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ such that, for each $x, y \in X$,

$$\rho[Tx, Ty] \leqq a_1\,\rho[x, y] + a_2\,\rho[x, Tx] + a_3\,\rho[y, Ty] + a_4\,\rho[x, Ty] + a_5\,\rho[y, Tx].$$

(13) (Ćirić [9], S. Massa [18]) There exists a constant $q \in [0, 1)$, such that, for each $x, y \in X$,

$$\rho[Tx, Ty] \leqq q \max\{\rho[x, y], \rho[x, Tx], \rho[y, Ty], \rho[x, Ty], \rho[y, Tx]\}.$$

Geometricaly:



$$\begin{array}{l}
(A) \Rightarrow (4) \not\Rightarrow (5) \Rightarrow (6) \Rightarrow (D) \\
\qquad\ \ \Downarrow \\
\qquad (6) \quad (7) \Rightarrow (8) \ \Rightarrow (10) \quad (9) \\
\qquad\qquad\ \ \Downarrow \qquad\qquad\qquad\quad \Downarrow \\
(7) \Leftrightarrow (8) \Rightarrow (9) \not\Rightarrow (11) \Rightarrow (12) \Leftrightarrow (13) \Rightarrow (B),\ (C),\ (T),\ (D).
\end{array}$$

Since conditions (A) and $(4)-(13)$ imply and the condition of generalized f-contraction, our Theorem 2 is a generalizations or Theorems of Massa [18], Ćirić [10], R. Kannan [15], S. Reich [20], Rhoades [21], Bianchini [7], Hardy-Rogers [12], Kurepa [17], Rakotch [19], Boyd and Wong [5], I. Rus [22], and others.

The following example shows that a generalized f-contraction need not be a condition (A) and $(4)-(13)$.

Example 1. Let $X=[0,+\infty)$ and define $T:X\to X$ by $Tx=x(1+x)^{-1}$, and distance function ρ is the ordinary euclidiean distance on the line. The mapping T is a generalized f-contraction already for mapping $f:(\mathbf{R}^0_+)^5\to\mathbf{R}^0_+$, let be define as

$$f(t_1, t_2, t_3, t_4, t_5) \overset{\text{def}}{=} t_1(1+t_1)^{-1}, \quad (t_1 \geqq 0).$$

Then it is easy to verify that f satisfies all the conditions of Theorem 2. Furthermore, for any $x, y \in X$

$$\rho[Tx, Ty] = \frac{|x-y|}{1+x+y+xy} \leqq \frac{|x-y|}{1+|x-y|} =$$

$$= f(\rho[x, y],\ \rho[x, Tx],\ \rho[y, Ty],\ \rho[x, Ty],\ \rho[y, Tx]).$$

Thus (C) holds. Since X is T-orbitally complete, it follows by Theorem 2 that T has a unique fixed point — it is a point 0. However, T does not satisfy (A) and $(4)-(13)$, for otherwise there is a $q<1$ such that for all $x\in X$:

$$(14) \qquad \rho[To, Tx] = \frac{x}{1+x} \leqq q \max\left\{0,\ \frac{x^2}{1+x},\ \ x,\ \frac{x}{1+x},\ \ x\right\}.$$

Since for any $x \in X$, $x^2(1+x)^{-1} \leqq x$, it follows by (14) that for each $x>0$, $x(1+x)^{-1} \leqq qx$ that is $(1+x)^{-1} < q$ for each $x>0$.

This is clearly impossible. Thus, T does not satisfy (A) and (4)–(13) for any value of $q<1$. Therefore, S. Massa, Ćirić, Kannan, Reich, Rhoades B., Hardy-Rogers, Kurepa, Rakotch, Boyd and Wong, I. Rus and other authors results is in fact a special case of Theorem 2.

The next result follows easily from the above Theorem.

Theorem 3. *Let T be a mapping of a metric space X into itself and let X be T-orbitally complete.* 1) *If there exists a positive integer k such that the iteration T^k is a f-contraction, then*

(a) *T has a unique fixed point ξ in X;*

(b) $\lim T^n x = \xi$, *and*

(c) $\rho[T^n x, \xi] \leqq \lambda^m (1-\lambda)^{-1} \mathcal{M}$ *for every* $x \in X$,

where $\mathcal{M} = \max\{\rho[T^i x, T^{i+k} x] : i = 0, 1, \ldots, k-1\}$ *and $m = E(n)$ is the greatest integer not exceeding n/k.*

2) *If there exist a positive integer k such that the iteration T^k is a generalized f-contraction, then T has a unique fixed point ξ in X and* $\lim T^n x = \xi$.

1.2. Definition. *A mapping T of a space X into itself is said to be orbitally continuous if $u \in X$ and such that $u = \lim\limits_{i \to \infty} T^{n_i} x$ for some $x \in X$, then $Tu = \lim\limits_{i \to \infty} T T^{n_i} x$.*

Generalized f-contractions and f-contractions need not be continuous, but are such that fixed point theorems may be proved without assumption of an orbitally continuity. The following theorem shows that every generalized f-contraction and f-contraction is orbitally continuous in the sense of the definition.

Theorem 4. *Let $T: X \to X$ be a f-contraction or generalized f-contraction mappings of a metric space X into itself. If $u \in X$ is such that $u = \lim\limits_{i \to \infty} T^{n_i} x$ for some $x \in X$, then $Tu = \lim\limits_{i \to \infty} T^{n_i+1} x$.*

1.3. The purpose of the present section is to prove one Theorem in which we have omitted the completeness of the space from each. We have obtained the same conclusion as in Banach's Theorem but with different sufficient conditions.

Theorem 5. *Let X be a metric space with ρ as metric. Let T be a map of X into itself such that:*

(a) *T is an f-contraction or generalized f-contraction;*

(b) *T is continuous at a point* $\xi \in X$, *and*

(c) *There exists a point* $x \in X$ *such that the sequence of iterates* $(T^n x)$ *has a subsequence* $(T^{n_i} x)$ *converging to* ξ.

Then ξ *is the unique fixed point of T.*

Remark: The proof of this theorem is very similar to that of M. Edelstein in his paper [11]. If we compare this theorem with Theorem 1, 2 we see that we have omitted the completeness of the space and instead, we have assumed conditions (*b*) and (*c*). The conditions (*b*) and (*c*) in Theorem 3 together do not guarantee the completeness of the space. The easy example in support of this is the following:

Example 2. Let $X=[0, 1)$, $Tx=x/3$ and the distance function ρ is the ordinary euclidean distance on the line.

2. Proofs of Theorems

Proof of Lemma 1. (a) Let $x \in X$ be arbitrary, n be any positive integer, and i and j satisfy the condition of Lemma. Then $T^{i-1}x, T^i x, T^{j-1}x, T^j x \in \mathcal{O}(x, n)$ and since T is a f-contraction and from Pıoposition 1 for sequence $x_{i,j} \overset{\text{def}}{=} \rho[T^i x, T^j x]$ we have

$$\rho[T^i x, T^j x] = \rho[T(T^{i-1}x), T(T^{j-1}x)] \leqq f(\alpha_1 \rho[T^{i-1}x, T^{j-1}x],$$

$$\alpha_2 \rho[T^{i-1}x, T^i x],\ \alpha_3 \rho[T^{j-1}x, T^j x],\ \alpha_4 \rho[T^{i-1}x, T^j x],\ \alpha_5 \rho[T^i x, T^{j-1}])$$

$$\leqq \mathscr{L}\, \theta^{\max\{i,j\}} \leqq \lambda \sigma[\mathcal{O}(x, n)],\ \mathscr{L} = \max_{i=1,2,\ldots,k} \{x_{i,p}\theta^{-i}\},\ \theta \in (0, 1),$$

which proves the assertion (a) of Lemma, and the following corollary (*b*)

$$(\forall x \in X)\ (\exists k \leqq n)\ \rho[x, T^k x] = \sigma[\mathcal{O}(x, n)].$$

(c) Let $x \in X$ be arbitrary. Since $\sigma[\mathcal{O}(x, 1)] \leqq \sigma[\mathcal{O}(x, 2)] \leqq \ldots$, we see that $\sigma[\mathcal{O}(x, \infty)] = \sup\{\sigma[\mathcal{O}(x, n)] : n \in N\}$. Then let n be any positive integer. From the corollary to the previous Lemma, there exists $T^k x \in \mathcal{O}(x, n)$ $(k \leqq n)$ such that $\rho[x, T^k x] = \sigma[\mathcal{O}(x, n)]$. Applying the t iangle inequality and assertion (a) of Lemma, we get

$$\rho[x, T^k x] \leqq \rho[x, Tx] + \rho[Tx, T^k x]$$

$$\leqq \rho[x, Tx] + \lambda \sigma[\mathcal{O}(x, n)] = \rho[x, Tx] + \rho[x, T^k x].$$

Therefore,

$$\sigma[\mathcal{O}(x, n)] = \rho[x, T^k x] \leqq (1-\lambda)^{-1} \rho[x, Tx].$$

Since n is arbitrary, the proof is complete.

Proof of Theorem 1. Let x be an arbitrary point of X. We can then show that the sequence of iterates $(T^n x)$ is a Cauchy sequence. Let n and m $(n<m)$ be any positive integers. Since T is a f-contraction, it follows by Lemma 1 that

$$\rho[T^n x, T^m x]=\rho[T(T^{n-1}x), T^{m-n+1}(T^{n-1}x)]\leqq\lambda\sigma[O(T^{n-1}x, m-n+1)].$$

According to the above Lemma 1, there exists an integer k_1, $n\leqq k_1\leqq n-m+1$, such that

$$\rho[T^{n-1}x, T^{k_1}(T^{n-1}x)]=\sigma[O(T^{n-1}x, m-n+1)].$$

Since $T^{n-1}x=TT^{n-2}x$, $T^{k_1}(T^{n-1}x)=T^{k_1+1}(T^{n-2}x)$, applying now Lemma 1 ($a$) to $\rho[TT^{n-2}x, T^{k_1+1}T^{n-2}x]$ we obtain

$$\rho[T^{n-1}x, T^{k_1}T^{n-1}x]\leqq\lambda\sigma[O(T^{n-2}x, k_1+1)]\leqq\lambda\sigma[O(T^{n-2}x, m-n+2)].$$

Therefore, we have

$$\rho[T^n x, T^m x]\leqq\lambda\sigma[O(T^{n-1}x, m-n+1)]\leqq\lambda^2\sigma(T^{n-2}x, m-n+2)].$$

Proceeding in this manner, we obtain

$$\rho[T^n x, T^m x]\leqq\lambda\sigma[O(T^{n-1}x, m-n+1)]\leqq\cdots\leqq\lambda^n\sigma[O(x, m)].$$

Then it follows by Lemma 1. that

$$\rho[T^n x, T^m x]\leqq\lambda^n(1-\lambda)^{-1}\rho[x, Tx], \tag{15}$$

which proves that the sequence $(T^n x)$ is a Cauchy sequence. Since X is X is T-orbitally complete, $(T^n x)$ has a limit ξ in X, $(\xi=\lim x_n=\lim Tx_{n-1}=\lim T^n x)$. To prove that $T\xi=\xi$, consider the following inequalities:

$$\rho[x_n, T\xi]=\rho[Tx_{n-1}, T\xi]\leqq f(\alpha_1\rho[x_{n-1}, \xi], \alpha_2\rho[x_{n-1}, x_n], \alpha_3\rho[\xi, T\xi], \alpha_4\rho[x_{n-1}, T\xi], \alpha_5\rho[\xi, x_n]) \tag{15.1}$$

and from Lemma 1 we have

$$\rho[x_n, T\xi]\leqq\mathscr{L}\theta^n \quad (\theta\in(0, 1), n\in N).$$

Letting $n\to\infty$ one gets $\rho[\xi, T\xi]=0$. Hence ξ is a fixed point under T. The uniqueness follows by f-contractivity of T, where, if δ is an element of X such that $T(\delta)=\delta$, then from Lemma 1

$$r=\rho[\xi, \delta]\leqq f(\alpha_1 r, \alpha_2 r, \alpha_3 r, \alpha_4 r, \alpha_5 r)\Rightarrow r=0$$

So we proved (a) and (b), as x was arbitrary. Letting now m tend to infinity in (15), we obtain the inequality (c). The proof is complete.

Proof of Proposition 2. Let $t_n=\max\{x_n, x_{n+1}, \ldots, x_{n+k-1}\}$, then $t_n\geqq\alpha$ or $t_n<\alpha$. Now if $t_n\geqq\alpha$, by the properties of f (semihomogeneous order α and monotonicaly)

$$x_{n+k}\leqq f(x_n, x_{n+1}, \ldots, x_{n+k})\leqq f(t_n, t_n, \ldots, t_n)$$
$$\leqq t_n f(1, 1, \ldots, 1)$$

Applying Proposition 1 we obtain $x_n \leqq \mathscr{L}\theta^n (n\in N,\ \theta\in(0,1))$. Hence for $n\to\infty$, $x_n\to 0$. Contradiction. Meanwhile, for $t_n<\alpha$ we obtain.

$$x_{n+k}\leqq f(x_n, x_{n+1}, \ldots, x_{n+k})\leqq f(t_n, t_n, \ldots, t_n)<t_n.$$

Therefore, for all $n\in N$ is $t_{n+1}\leqq t_n$, and $t_n=x_{n+k-s(n)}$ for some $s(n)$ between 1 and k, where $x_{n+k}<t_n (n\in N)$. Therefore either t_n tends to a finite limit q. Therefore $\limsup x_n\leqq q$. We now prove that $\liminf x_n\geqq q$. Putting $n=m+k+1$ and $s(n)=2k+1-i$ we obtain

$$x_{m+i}=t_{m+k+1}\geqq q. \tag{16}$$

Now, by (16) and the monotonicity of f,

$$x_{m+i}\leqq f(x_m, x_{m+1}, \ldots, x_{m+i})\leqq f(t_m, t_m, \ldots, t_m), \tag{17}$$

where t_m is in every place except the sth, where there is x_n. Here i is a function of m and its values can be $1, 2, \ldots, k$. Now if $\liminf x_m=q-2\varepsilon<q$, then there exists a strictly increasing subsequence $\{m_e\}$ of the positive integers such that $x_m<q-\varepsilon$ for $e=1, 2, 3, \ldots$ Moreover, we may choose the subsequence so that each m_e corresponds to the same value of i in (17). Hence, from (16) and (17), $q\leqq f(t_{m_e}, \ldots, t_{m_e})$ where, for all $e\geqq 1$, $q-\varepsilon$ occurs in the same sth place. Letting $e\to\infty$, we deduce from the condition of f that $q\leqq f(q, \ldots, q)$, which contradicts. Hence $q\leqq\liminf x_n$ and this, together with the result $\limsup x_n\leqq q$, shows that

$$\lim x_n=q\ (q\leqq f(q, \ldots, q)<q\Rightarrow q=0).$$

Proof of Theorem 2. For each $n\in N$, let $\Delta_n=\sup\{\rho[T^p x, T^q x]: p, q\geqq n\}$. Since Δ_n is nonincreasing sequence in R_+, there is a $\Delta\geqq 0$ such that $\Delta_n\to\Delta\ (n\to\infty)$. We claim that $\Delta=0$. If $\Delta>0$ then for any $p, q\in N$,

$$\rho[T^p x, T^q x]\leqq f(\rho[T^{p-1}x, T^{q-1}x], \rho[T^{p-1}x, T^p x], \rho[T^{q-1}x, T^q x], \rho[T^{p-1}x, T^q x], \rho[T^{q-1}x, T^p x]).$$

Therefore, if $p, q\geqq n$, it follows that

$$\Delta_n\leqq f(\Delta_{n-1}, \Delta_{n-1}, \Delta_{n-1}, \Delta_{n-1}, \Delta_{n-1})$$

and hence applying Proposition 2 to the sequence (Δ_n) we obtain $\Delta=0$. This, implies that $(T^n x)$ is a Cauchy sequence in X and hence, by completeness, there is a $\xi\in X$ such that $T^n x\to\xi\ (n\to\infty)$. Now since (17.1.):

$$\rho[T\xi, T^{n+1}x]\leqq f(\rho[\xi, T^n x], \rho[\xi, T\xi], \rho[T^n x, T^{n+1}x], \rho[\xi, T^{n+1}x], \rho[T^n x, T\xi]).$$

Therefore, as $n\to\infty$ the above inequality yields

$$r=\rho[T\xi, \xi]\leqq f(0, r, 0, 0, r)\leqq f(r, r, r, r, r)$$

and hence applying Proposition 2 we obtain thus $T\xi=\xi$.

To prove uniqueness, suppose theıe is a $u \neq \xi$ for which $Tu = u$, and $T\xi = \xi$. Then

$$r = \rho[u, \xi] = \rho[Tu, T\xi] \leqq f(r, 0, 0, r, r) \leqq f(r, r, r, r, r) < r,$$

contradicting $r > 0$. Thus $\xi = u$.

Proof of Theorem 3. Since T^k has a unique fixed point ξ and $T^k(T\xi) = T(T^k\xi) = T\xi$, it follows that $T\xi = \xi$. Its uniqueness is obvious.

To show (c), let n be any integer. Then $n = mk + j$, $0 \leqq j < k$, $m \geqq 0$, and for every $x \in X$, $T^n x = (T^k)^m T^j x$. Since T^k is a f-contraction, it follows by part (c) of Theorem 1 that

$$\rho[T^n x, \xi] \leqq \lambda^m (1-\lambda)^{-1} \rho[T^j x, T^k T^j x]$$
$$\leqq \lambda^m (1-\lambda)^{-1} \max\{\rho[T^i x, T^k T^i x] : i = 0, 1, \ldots, k-1\}$$

which proves (c) and, hence, (b). The proof is complete.

Here we utilize the following interesting proposition of Adamović:

Proposition 3 (Adamović [1]) *If T is a mapping of a nonempty set X into itself and, for some natural number n, the iterate T^n possesses a unique fixed point, then T has a unique fixed point, too.*

Proof of Theorem 4. Let u and x in X be such that $u = \lim_{i\to\infty} T^{n_i} x$. Consider the sequence $\{T^n x : n \in N\}$ which contains the sequence $\{T^{n_i} x : i \in n\}$ as a subsequence. Since T is a (generalized) f-contraction, it follows that the sequence $\{T^n x : n \in N\}$ must be Cauchy, as it was shown in the first part of the proof of the theorems 1 and 2. Since $\{T^n x : n \in N\}$ is a Cauchy sequence and contains a subsequence $\{T^{n_i} x \cdot i \in N\}$ such that $\lim_{i\to\infty} T^{n_i} x = u$, it follows that $\lim_{n\to\infty} T^n x = u$. Then, from (15.1.) or (17.1) one has $\lim \rho[Tu, TT^n x] = 0$, i. e. $Tu = \lim T^{n+1} x$ which implies $Tu = \lim_{i\to\infty} T^{n_i+1} x$. This completes the proof.

Proof of Theorem 5. Continuity at ξ of T implies that $(T^{n_i+1} x)$ converges to $T(\xi)$. Suppose $\xi \neq T(\xi)$. We consider two open discs $S_1 = S_1(\xi, \eta)$ and $S_2 = S_2(T(\xi), \eta)$ centered at ξ and $T(\xi)$, respectively, and of radius $\eta > 0$, where $\eta < 3^{-1} \rho[\xi, T(\xi)]$. Since $(T^{n_i} x)$ converges to ξ and $(T^{n_i+1} x)$ converges to $T(\xi)$, there exists a positive integer N_1 such that $i > N_1$ implies $T^{n_i} x \in S_1$ and $T^{n_i+1} x \in S_2$. Hence,

$$\rho[T^{n_i} x, T^{n_i+1} x] > \eta, \qquad (i > N_1). \tag{18}$$

On the other hand, by condition (a), we have

$$\rho[T^{n_i+1} x, T^{n_i+2} x] \leqq f(\alpha_1 \rho[T^{n_i} x, T^{n_i+1} x], \alpha_2 \rho[T^{n_i} x, T^{n_i+1} x],$$
$$\alpha_3 \rho[T^{n_i+1} x, T^{n_i+2} x], \alpha_4 \rho[T^{n_i} x, T^{n_i+2} x], \alpha_5 \rho[T^{n_i+1} x, T^{n_i+1} x])$$

For $e>j>N_1$ and applying now Lemma 1 we have

$$\rho[T^{n_e}x, T^{n_e+1}x] \leqq \lambda\sigma[O(x, n_e+1)] \leqq \lambda^2\sigma[O(x, n_e+2)] \leqq \cdots$$
$$\leqq \lambda^{n_j}\sigma[O(x, n_e+n_j)] \leqq \lambda^{n_e-n_j}(1-\lambda)^{-1}\rho[x, Tx].$$

However, this last expression approaches 0 as e approaches ∞, and we would get a result contradicting (18). Hence $T(\xi)=\xi$. Hence, ξ is a fixed point of T. If δ is an element of X such that $T(\delta)=\delta$, then

$$r=\rho[\xi, \delta]=\rho[T\xi, T\delta] \leqq f(\alpha_1\rho[\xi, \delta], 0, 0, \alpha_4\rho[\xi, \delta], \alpha_5\rho[\xi, \delta])$$
$$\leqq f(\alpha_1 r, \alpha_2 r, \alpha_3 r, \alpha_4 r, \alpha_5 r) \Rightarrow r=0.$$

Hence $\xi=\delta$ and the theorem is proved for f-contractions.

One can prove in the same manner the part of this proposition concerning generalized f contractions.

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