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CURVATURE TENSOR OF PSEUDO METRIC SEMI-SYMMETRIC CONNEXIONS IN AN ALMOST CONTACT METRIC MANIFOLD

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1. Preliminaries: *A* C^∞ manifold M of dimension $n(=2m+1)$ is said to be an almost contact metric manifold if it admits a (1, 1) tensor field F, a vector field t, a 1-form A and a Riemannian metric g satisfying

$$(1.1)\qquad \text{(a) } A(t)=1, \qquad \text{(b) } F^2(X)=-X+A(X)t$$
$$\text{(c) } g(\overline{X}, \overline{Y})=g(X, Y)-A(X)A(Y), \quad F(X)\overset{\text{def}}{=\!=}(\overline{X})$$

for any vector fields X and Y.

In 1975 Prvanović [1] introduced pseudo metric semi symmetric connexions on a Riemannian manifold and studied some of their interesting properties. Following her definitions and adopting them in the setting of almost contact metric manifold we construct two connexions

$$(1.2)\qquad \overset{3}{\nabla}(AY)=A(\overset{1}{\nabla}Y)+(\overset{2}{\nabla}A)(Y), \text{ and}$$

$$(1.3)\qquad \overset{4}{\nabla}(AY)=A(\overset{2}{\nabla}Y)+(\overset{1}{\nabla}A)(Y),$$

in terms of the connexions

$$(1.4)\qquad \overset{1}{\nabla}_X Y=D_X Y+A(X)Y-g(X, Y)t, \text{ and}$$

$$(1.5)\qquad \overset{2}{\nabla}_X Y=D_X Y+A(Y)X-g(X, Y)t,$$

where D is the Riemannian connexion and $\overset{2}{\nabla}$ is the semi-symmetric metric connexion defined in [2], [3]. As shown in [1], $\overset{3}{\nabla}$ and $\overset{4}{\nabla}$ are the pseudo metric semi-symmetric connexions, i.e.

$$\overset{3}{\nabla}_k g_{ij}=0, \qquad \overset{4}{\nabla}_k g^{ij}=0,$$

but

$$\overset{4}{\nabla}_k g_{ij} = \overset{1}{\nabla}_k g_{ij} = -2A_k g_{ij} + A_i g_{jk} + A_j g_{ik},$$

and

$$\overset{3}{\nabla}_k g^{ij} = \overset{1}{\nabla}_k g^{ij} = 2A_k g^{ij} - t^i \delta_k^j - t^j \delta_k^i,$$

where A_i and t^i are the components of the form A and of the vector field t with respect to local co-ordinates. The purpose of the present paper is to study such connexions in an almost contact metric manifold.

2. Curvature tensor of the connexions $\overset{3}{\nabla}$ and $\overset{4}{\nabla}$.

Let K and $\underset{3}{R}$ denote the curvature tensors corresponding to the connexions D and $\overset{3}{\nabla}$ respectively. Then we find that in an almost contact metric manifold they are related by ([1]):

$$\begin{aligned}\underset{3}{R}(X, Y, Z) &= K(X, Y, Z) + \alpha(X, Y)Z - \alpha(Y, Z)X + g(X, Z)\beta Y \\ &\quad - g(Y, Z)\beta X - g(X, Z)Y + g(Y, Z)X,\end{aligned} \tag{2.1}$$

where α is a tensor field of type (0,2) defined by

$$\alpha(X, Y) = (D_X A)(Y) - A(X)A(Y) + g(X, Y) \tag{2.2}$$

and β is a tensor field of type (1,1) defined by

$$g(\beta(X), Y) = \alpha(X, Y). \tag{2.3}$$

Let us put

$$\underset{3}{'''\rho}(X, Y) \overset{\text{def}}{=} (C_3^1 \underset{3}{R})(X, Y), \tag{2.4}$$

where C_3^1 stands for contraction in the third slot. Then (2.1) gives

$$\alpha(X, Y) = \frac{1}{n-1} \underset{3}{'''\rho}(X, Y) \tag{2.5}$$

and

$$\beta(X) = \frac{1}{n-1} \underset{3}{'''R}(X), \tag{2.6}$$

where

$$g(\underset{3}{'''R}(X), Y) = \underset{3}{'''\rho}(X, Y). \tag{2.7}$$

Substituting (2.5) and (2.6) in (2.1), we get

$$\begin{aligned}\underset{3}{R}(X, Y, Z) - \frac{1}{n-1}\Big[&\underset{3}{'''\rho}(X, Y)Z - \underset{3}{'''\rho}(Y, Z)X + \\ &- g(X, Z)\underset{3}{'''R}(Y) - g(Y, Z)\underset{3}{'''R}(X)\Big] \\ = K(X, Y, Z) &+ g(Y, Z)X - g(X, Z)Y.\end{aligned} \tag{2.8}$$

Hence we have the result:

Theorem (2.1). *If an almost contact metric manifold M admits a pseudo-metric semi-symmetric connexion* $\overset{3}{\nabla}$ *whose curvature tensor* $\underset{3}{R}(Y, Y, Z, W)$ *vanishes, then the Riemannian curvature of M at* $m \in M$ *is independent of the directions X, Y, where X and Y are unit tangent vectors at* $m \in M$, *and is in fact equal to* -1.

Let $\underset{4}{R}$ be the curvature tensor with respect to the connexion $\overset{4}{\nabla}$. Then we see that in an almost contact metric manifold $\underset{4}{R}$ and K are related by ([1]):

$$(2.9)\qquad \begin{aligned}\underset{4}{R}(X, Y, Z) = {} & K(X, Y, Z) + \gamma(X, Y)Z - \gamma(Y, Z)X + g(X, Z)\,G(Y)\\ & - g(Y, Z)\,G(X) - \delta(X, Y)Z + \delta(X, Z)Y\\ & - g(X, Z)\,D(Y) + g(Y, Z)\,D(X),\end{aligned}$$

where

$$\gamma(X, Y) = (D_X A)(Y) \quad \text{and} \quad \delta(X, Y) = A(X)\,A(Y) - g(X, Y),$$

and G and D are the tensor fields of type (1,1) defined by

$$g(GX, Y) = \gamma(X, Y) \quad g(DX, Y) = \delta(X, Y).$$

Let us put

$$'\underset{4}{\rho}(Y, Z) \overset{\text{def}}{=} (C_1^1 \underset{4}{R})(Y, Z); \quad 'K(Y, Z) \overset{\text{def}}{=} (C_1^1 K)(Y, Z),$$

$$''\underset{4}{\rho}(X, Z) \overset{\text{def}}{=} (C_2^1 \underset{4}{R})(X, Z); \quad ''K(X, Z) \overset{\text{def}}{=} (C_3^1 K)(X, Z),$$

$$'''\underset{4}{\rho}(X, Y) \overset{\text{def}}{=} (C_3^1 \underset{4}{R})(X, Y), \quad '''K(X, Y) \overset{\text{def}}{=} (C_3^1 K)(X, Y) = 0,$$

$$'\underset{4}{r} = '\underset{4}{\rho}(X_i, X_i), \quad ''\underset{4}{r} = ''\underset{4}{\rho}(X_i, X_i),$$

$$'''\underset{4}{r} = '''\underset{4}{\rho}(X_i, X_i), \quad k = 'K(X_i, X_i),$$

X_i being n orthonormal vectors. Then (2.9) gives

$$(2.10)\qquad \begin{aligned}'\underset{4}{\rho}(Y, Z) = {} & K(Y, Z) + \gamma(Z, Y) - (n-1)\,\gamma(Y, Z) - \delta(Z, Y)\\ & - (G - D)\,g(Y, Z),\end{aligned}$$

$$(2.11)\qquad ''\underset{4}{\rho}(X, Z) = ''K(X, Z) + n\delta(X, Z) - \gamma(X, Z) + (G - D)\,g(X, Z),$$

and

$$(2.12)\qquad '''\underset{4}{\rho}(X, Y) = (n-1)\,[(X, Y) - \delta(X, Y)]$$

From (2.11) and (2.12), we get

$$G=\frac{1}{n-1}(''\underset{4}{r}+k), \text{ and}$$

$$D=\frac{1}{n-1}(''\underset{4}{r}-'''\underset{4}{r}+k),$$

and hence

$$G-D=\frac{1}{n-1}\,'''\underset{4}{r}. \tag{2.13}$$

From (2.10) and (2.12), we find that

$$\gamma(X, Y)=\frac{1}{n-1}\Big[{}'K(X, Y)+\frac{1}{n-1}\,'''\underset{4}{\rho}(X, Y)- -{}'\underset{4}{\rho}(X, Y)-\frac{1}{n-1}g(X, Y)'''\underset{4}{r}\Big], \tag{2.14}$$

and

$$\delta(X, Y)=\frac{1}{n-1}\Big[{}'K(X, Y)+\frac{2-n}{n-1}\,'''\underset{4}{\rho}(X, Y)- -{}'\underset{4}{\rho}(X, Y)-\frac{1}{n-1}g(X, Y)\,'''\underset{4}{r}\Big]. \tag{2.15}$$

Then in consequence of (2.13), (2.14) and (2.15) equation (2.9) takes the form

$$\begin{aligned}\underset{4}{R}(X, Y, Z)&-\frac{1}{(n-1)}\Big[{}'''\underset{4}{\rho}(X, Y)Z+g(X, Z)\,'''\underset{4}{R}(Y)-g(Y, Z)\,'''\underset{4}{R}(X)\Big]\\ &-\frac{1}{n-1}\Big[{}'\underset{4}{\rho}(Y, Z)-\frac{1}{n-1}\,'''\underset{4}{\rho}(Y, Z)+\frac{1}{n-1}g(Y, Z)\,'''\underset{4}{r}\Big]X\\ &+\frac{1}{n-1}\Big[{}'\underset{4}{\rho}(X, Z)-\frac{2-n}{n-1}\,'''\underset{4}{\rho}(X, Z)+\frac{1}{n-1}g(X, Z)\,'''\underset{4}{r}\Big]Y\\ &=K(X, Y, Z)-\frac{1}{n-1}[{}'K(Y, Z)X-{}'K(X, Z)Y].\end{aligned} \tag{2.16}$$

We now assume that $\underset{4}{R}(X, Y, Z, W)=0$. Then (2.16) becomes

$$K(X, Y, Z, W)=\frac{1}{n-1}\{{}'K(Y, Z)g(X, W)-{}'K(X, Z)g(Y, W)\} \tag{2.17}$$

Contracting in Y, Z, we get

$${}'K(X, W)=\frac{r}{n}g(X, W) \tag{2.18}$$

which shows that the manifold then is an *Einstein space*. From (2.17) and (2.18) it follows that Riemannian curvature is constant. Hence we conclude:

Theorem (2.2). *If an almost contact metric manifold M admits a pseudo-metric semi-symmetric connexion* $\overset{4}{\nabla}$ *whose curvature tensor* $\underset{4}{R}(X, Y, Z, W)$ *vanishes then the Riemannian curvature of M at $m \in M$ is independent of the directions X, Y where X and Y are unit tangent vectors at a point $m \in M$, (and is therefore $r/n(n-1)$).*

As shown in [1] the following are immediate consequences of (2.16):

Corollary (2.3). *If an almost contact metric manifold admits a pseudo--metric semi-symmetric connexion* $\overset{4}{\nabla}$ *whose Ricci tensors* $'\underset{4}{\rho}(X, Y)$ *and* $'''\underset{4}{\rho}(X, Y)$ *vanish then the curvature tensor* $\underset{4}{R}(X, Y, Z)$ *of* $\overset{4}{\nabla}$ *is equal to the projective curvature tensor of the Riemannian connexion*, (cf. [1]).

Corollary (2.4). *If the curvature tensor* $\underset{4}{R}(X, Y. Z)$ *of the pseudo--metric semi symmetric connexion* $\overset{4}{\nabla}$ *vanishes, then the projective curvature tensor of the Riemannian connexion also vanishes.*

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