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THE CHARACTERIZATION OF SERIES AMENABLE TO RATIO TESTS

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1. Introduction

Among the most popular tests of convergence of series of positive terms are the simple ratio test and its refinements in their limit forms. It is therefore of some interest to observe that the existence of the limit in the test criterion largely specifies the possible forms of the terms of the series.

Let $\mathcal{T}=\{t_n\}_{n=1}^{\infty}$ be a sequence of positive numbers and set $T=\sum_{n=1}^{\infty} t_n<\infty$. A general criterion for determining whether $T<\infty$, or otherwise, is Kummer's test [9, p. 311], [11, p. 106]. One form of this states that if $\{c_n\}$ is a positive term sequence with $\sum c_n^{-1}=\infty$, then $T<\infty$ (resp. $=\infty$) if $K_n=c_n t_n/t_{n+1}—c_{n+1}\to K>0$ (resp. <0). If $K=0$ we cannot make a decision either way. An interesting alternative form [4], [10] is that $T<\infty$ iff there is a positive term sequence $\{c_n\}$ such that $\lim\limits_{n\to\infty}(c_n t_n/t_{n+1}—c_{n+1})>0$. It was essentially this form that was first given by Kummer [4, p. 38]. Kummer's test includes the limit forms of the well known tests such as the D'Alembert-Waring-Cauchy ratio test [8, p. 465] ($c_n\equiv 1$) and Raabe's test ($c_n=n$).

These latter tests can be taken as the first two of an infinite sequence of tests. Indeed, let $\log_o x=x$, $\log_{k+1} x=\log(\log_k x)$ $(k=1, 2, \ldots)$, $D_{-1}(\mathcal{T}, n)=t_n/t_{n+1}$; and for $k=0, 1, \ldots,$ and n sufficiently large,

$$D_k(\mathcal{T}, n)=(\log_k n)(D_{k-1}(\mathcal{T}, n)—1).$$

If for some k, $\lim\limits_{n\to\infty} D_k(\mathcal{T}, n)>1$ (resp. <1) then $T<\infty$ (resp. $T=\infty$). The cases $k=—1$, 0, and 1 yield the ratio, Raabe's and Bertrand's tests, respectively. If for some k, $D_k(\mathcal{T}, n)\to 1$ then a decision may be possible on the basis of $D_{k'}(\mathcal{T}, n)$ for some $k'>k$. Thus we have a sequence of increasingly fine tests which may be applied to a given series.

In this paper we shall determine the classes of series for which these tests are applicable. Specifically, we shall establish the nature of the sequence

$\mathcal{J}$ for which $\lim_{n\to\infty} D_k(\mathcal{J}, n)$ exists as a finite number. This characterization is carried out using recent results from the theory of regularly varying functions and sequences.

In § 3 we prove the main result which shows (roughly) that in order for $D_k(\mathcal{J}, n)\to\rho$ $(n\to\infty)$, t_n must be asymptotically proportional to

$$s_n=[\Lambda_k(n)(\log_k n)^{\rho} M_k(\log_k(n))]^{-1}$$

where we write $\Lambda_{-1}(n)\equiv 1$ and $\Lambda_k(n)=\prod_{j=0}^{k}\log_j n\,(k=0, 1, \ldots)$ (arguments of functions are always assumed sufficiently large for the functions to be well-defined) and $M_k(x)$ belongs to a special class of functions. This result is closely related to the so called logarithmic scale which consists of those series whose terms are of the form above with $M_k(x)\equiv 1$. For each real ρ this gives an indexed family of series which diverge or converge more slowly with increasing k. The collection of these families provides a set of series which are useful for comparison purposes. We refer the reader to [9, p. 278] for accounts of this topic.

Our representation theorem provides estimates of the rate of convergence or divergence of series amenable to ratio tests and also provides some insight to generalisations of Gauss' test. These matters are discussed in § 3. See [10] for some related results.

2. Positive term series.

A positive measurable function $R(x)$, $0<x<\infty$, is said to be regularly varying (at infinity) with index ρ, $-\infty<\rho<\infty$ or simply ρ-varying if $\lim_{x\to\infty} R(\lambda x)/R(x)=\lambda^{\rho}$ for $\lambda>0$. If $\rho=0$, $R(x)$ is called slowly varying (S. V.). A ρ-varying function can be written in the form $R(x)=x^{\rho}L(x)$ where $L(x)$ is S. V. A positive measurable function $g(x)$, $0\leqslant x<\infty$ is said to be of moderate growth (M. G.) if $\lim_{x\to\infty} g(\lambda+x)/g(x)=1$ for each real λ. This is equivalent to the requirement that $g(\log x)$ be S. V. Finally we say that $g(x)$ is ρ-rapid if $g(\log x)$ is ρ-varying and hence is of the form $e^{\rho x}L(e^x)$ where $L(x)$ is S. V. See [2], [5] and [6] for details.

Our first result characterises those series amenable to the D'Alembert-Waring-Cauchy test.

Theorem 1.

(1) $$\lim_{n\to\infty} D_{-1}(\mathcal{J}, n)=\lambda$$

exists and is finite and positive iff there is a ρ-rapidly varying function $g(x)$, such that $t_n=g(n)$. If (1) *holds then*

$$g(x)=\lambda^{-x}M(x)$$

where $M(x)$ is of M. G.

Proof. The direct part is obvious. To prove the converse, observe that the hypothesis gives $D_{-1}(\mathcal{J}, n)=\lambda+\varepsilon(n)$, where $\varepsilon(n)\to 0$ and this yields

$$t_n=t_0\lambda^{-n}\left[\prod_{m=1}^{n}(1+\lambda^{-1}\varepsilon(m))\right]^{-1}.$$

Denoting the product by P_n, we have

$$P_n=\exp\sum_{m=1}^{n}\log(1+\lambda^{-1}\varepsilon(m))..$$

Consider the function $\eta(x)=\int_1^x\log(1+\lambda^{-1}\varepsilon([1+y]))\,dy$. A change of variables yields

$$\eta(\log x)=\int_e^y y^{-1}\log(1+\lambda^{-1}\varepsilon([1+\log y]))\,dy$$

so $\exp\eta(\log x)$ is S. V., hence $M(x)=\exp\eta(x)$ is of M. G. The theorem now follows since $P_n=M(n)(1+\lambda^{-1}\varepsilon(1))$.

A sequence $\{c(n)\}$ of positive terms is called regularly varying if

$$\lim_{n\to\infty}c([\lambda n])/c(n)=\psi(\lambda),\qquad 0<\psi(\lambda)<\infty$$

for each $\lambda>0$. It can be shown that $\psi(\lambda)=\lambda^\rho$ for some finite ρ and that $R(x)=c([x])$ is ρ-varying. If $\rho=0$ we say the sequence is S. V. A slightly earlier attempt [7] at defining the notion of a regularly varying sequence eventually yielded the following theorem, which was the inspiration for the present work.

Theorem 2. *A sequence of positive numbers $\{c(n)\}$ is ρ-varying iff there is a sequence of positive numbers $\mathcal{J}=\{a(n)\}$ such that $c(n)/a(n)\to 1$ and $D_0(\mathcal{J}, n)\to-\rho\ (n\to\infty)$.*

This result can be loosely interpreted by saying that Raabe's test can be applied to the series $\sum t_n$ iff $\{t_n\}$ is a regularly varying sequence.

We now turn our attention to the quantities $D_k(\mathcal{J}, n)$ $(k\geqslant 0)$. First observe that if $D_0(\mathcal{J}, n)\to\rho$, then $\mathcal{J}$ is monotonically decreasing (resp. increasing) if $\rho>0$ (resp. $\rho<0$). In particular if $\lim_{n\to\infty}D_k(\mathcal{J}, n)$ is finite for $k\geqslant 1$, then $D_0(\mathcal{J}, n)\to 1$. Thus a convergence test involving the existence of $\lim_{n\to\infty}D_k(\mathcal{J}, n)$ $(k\geqslant 1)$ entails the monotonicity of the terms of the series. This remark generalises one made by Ney [10, p. 6].

We now consider the problem of characterising the sequences $\mathcal{J}=\{t_n\}$ such that $\{D_k(\mathcal{J}, n)\}$ converges to a finite limit for some $k\geqslant 0$. This requires some preliminary results.

Lemma 1. $\log_k(n+1)=\log_k n+(\Lambda_{k-1}(n))^{-1}(1+0(n^{-1}))$.

Proof. The proof is by induction on k. The lemma is clearly true for $k=1$. Assuming it is true for k, we have

$$\begin{aligned}\log_{k+1}(n+1) &= \log[\log_k n + (\Lambda_{k-1}(n))^{-1}(1+0(n^{-1}))] \\ &= \log\{(\log_k n)[1+(\Lambda_k(n))^{-1}(1+0(n^{-1}))]\} \\ &= \log_{k+1} n + (\Lambda_k(n))^{-1}(1+0(n^{-1})) + 0((\Lambda_k(n))^{-2}).\end{aligned}$$

Lemma 2. *Let* $T_k(n)=\Lambda_k(n)/n$. *Then for* $k\geqq 1$.

(i) $$T_k(n+1)=T_k(n)+n^{-1}\left[\sum_{j=2}^{k}\prod_{i=j}^{k}\log_i n + 1 + 0(n^{-1}\log_2 n\cdots\log_k n)\right];$$

(ii) $$\Lambda_k(n+1)=\Lambda_k(n)+\sum_{j=1}^{k}\prod_{i=j}^{k}\log_i n + 1 + 0(T_k(n)/n\log n);$$

and

(iii) $$\Lambda_k(n+1)/\Lambda_k(n)=1+\sum_{j=1}^{k} 1/\Lambda_j(n)+0(1/n^2\log n).$$

Proof. Assertion (i) implies (ii), which in turn implies (iii). Since $T_{k+1}(n+1)=\log_{k+1}(n+1)\,T_k(n+1)$, (i) is obtained from Lemma 1 and induction on k.

Theorem 3. *If* $k\geqq 0$ *and* $\{D_k(\mathcal{T}, n)\}$ *converges to a finite limit,* ρ, *then* $t_n=a(n)R(n)$ *where* $R(x)$ *is a regularly varying function of the form*

$$R(x)=[\Lambda_{k-1}(x)(\log_k x)^{\rho} L_k(x)]^{-1},$$

$L_k(x)$ *is S.V. and of the form*

$$L_k(x)=\exp\int_{\zeta}^{x}\frac{\delta(y)}{\Lambda_k(y)}dy$$

where $\delta(x)\to 0\ (x\to\infty)$, ζ *is such that the integrand is well defined and* $\lim_{x\to\infty} a(x)$ *is finite and positive. Conversely if* t_n *can be represented in this way, then* $\sum t_n$ *converges and diverges with a series,* $\sum s_n$, *where* $D_k(\mathcal{S}, n)\to\rho$, $\mathcal{S}=\{s_n\}$.

Proof. The case $k=0$ is just a restatement of Theorem 1 and the remarks following it, and we take it as proven. The proof for the case $k\geqq 1$ is an elaboration of some of the working in [7]. Expanding $D_k(\mathcal{T}, n)$ gives

$$D_k(\mathcal{T}, n)=\Lambda_k(n)t_n/t_{n+1}-\sum_{j=0}^{k}\prod_{i=j}^{k}\log_i n=\rho+o(1)$$

so that

$$t_n/t_{n+1}=1+\sum_{j=0}^{k-1}1/\Lambda_j(n)+(\rho+o(1))/\Lambda_k(n).$$

Writing $t_n = (\Lambda_{k-1}(n)\, l_k(n))^{-1}$ and using Lemma 2 (iii) we obtain

$$l_k(n+1)/l_k(n) = 1 + 0\,(1/n^2 \log n) + (\rho + o(1))/\Lambda_k(n)$$
$$= 1 + (\rho + \varepsilon(n))/\Lambda_k(n)$$

where $\varepsilon(n) \to 0\ (n \to \infty)$. For $n \geqslant N$ say, $|\,1 - l_k(n+1)\, l_k(n)\,| < 1$, so that, following [7], we obtain

$$\log l_k(n) - \log l_k(N) = -\sum_{m=N}^{n} \sum_{j=2}^{\infty} j^{-1}(1 - l_k(m+1)/l_k(m))^j$$

$$(2) \qquad + \sum_{m=N}^{n} \frac{\rho + \varepsilon(m)}{\Lambda_k(m)}\,.$$

However $|\,1 - l_k(m+1)/l_k(m)\,|^j \leqslant (\rho + m^{-1}\varepsilon(m))^j$ so the argument in [7] shows that the first sum at (2) converges as $n \to \infty$.

Consider

$$d_n = \sum_{m=N}^{n} (\Lambda_k(m))^{-1} - \log_{k+1} n.$$

Then $d_n - d_{n-1} = (\Lambda_k(n))^{-1} - (\log_{k+1} n - \log_{k+1}(n-1))$

$$= (\Lambda_k(n))^{-1} - \int_{n-1}^{n} (\Lambda_k(x))^{-1}\, dx \leqslant 0$$

and $d_n \geqslant \int_{N}^{n+1} (\Lambda_k(x))^{-1}\, dx - \log_{k+1} n$

$$= \log_{k+1}(n+1) - \log_{k+1} n - \log_{k+1} N \geqslant -M > -\infty$$

where we use Lemma 1 to obtain the penultimate inequality. Thus $\lim_{n\to\infty} d_n$ exists. We finally have

$$l_k(n) = b(n)\,(\log_k n)^\rho \exp \sum_{m=N}^{n} \frac{\varepsilon(m)}{\Lambda_k(m)}$$

where $\lim_{n\to\infty} b(n)$ is finite and positive. The sum is interpolated by the expression $\varepsilon(N)/\Lambda_k(N) + \int_N^x \varepsilon([y+1])/\Lambda_k([y+1])\, dy$. However if N is large enough then for $y \geqslant N$, $\Lambda_k(y) \geqslant y$ and Lemma 2 (ii) shows that $\Lambda_k(1+y) - \Lambda_k(y) = 0\,(\Lambda_k(y)/y)$ and hence it follows that the difference of the integral above and $\int_N^x \varepsilon([y+1])/\Lambda_k(y)\, dy$ converges as $x \to \infty$.

To prove the converse part, define s_n by

$$s_n = [\Lambda_{k-1}(n)\,(\log_k n)^\rho\, L_k(n)]^{-1}.$$

Lemma 1, the expression for $L_k(x)$ and an integral mean value theorem yield

$$s_n/s_{n+1}=\frac{\Lambda_{k-1}(n+1)}{\Lambda_{k-1}(n)}\left(1+\frac{\rho}{\Lambda_k(n)}+o\left(\frac{1}{\Lambda_{k+1}(n)}\right)\right).$$

Lemma 2 (iii) now gives

$$D_k(\mathcal{S}, n)=\Lambda_k(n)\left(s_n/s_{n+1}-\Lambda_{k-1}(n+1)/\Lambda_{k-1}(n)+0\left(\frac{1}{n^2\log n}\right)\right)$$

$$=\frac{\Lambda_{k-1}(n+1)}{\Lambda_{k-1}(n)}(\rho+o(1)\to\rho \quad (n\to\infty).$$

3. Some Consequence of Theorem 3.

(i) *Some convergence rates.* The remarks preceding Theorem 3 show that if $\lim_{n\to\infty} D_k(\mathcal{J}, n)$ exists for some $k\geqslant 1$, and is non-zero if $k=0$, then we could apply the integral test to the series $\sum t_n$. We way use this to deduce the rate of divergence or convergence of $\sum t_n$. Suppose that $\rho<1$, so that $\sum t_n=\infty$ and write $A=\lim_{x\to\infty} a(x)$ $(0<A<\infty)$. Then

$$\sum_{m=1}^{n} t_m\sim A\int_{\zeta}^{n} R(y)\,dy.$$

where $R(x)$ has the form given in Theorem 3.

Denoting the integral by $I(n)$ we have

$$I(n)=\int_{\log_k \zeta}^{\log_k n}\frac{dy}{y^{\rho}\, l_k(y)}$$

where $l_k(x)=L(e_k(x))$, $e_k(x)$ being the inverse function of $\log_k(x)$. It is easily seen that $e_k(x)$ is defined for all real x and indeed, is defined by $e_0(x)=x$ and $e_k(x)=\exp(e_{k-1}(x))$. This recursive expression yields

$$\frac{d}{dx}e_k(x)=\prod_{j=1}^{k} e_j(x).$$

Substituting $y=e_k(z)$ in the integral defining $l_k(x)$ gives

$$l_k(x)=\exp\int_{\log_k \zeta}^{x}\frac{\delta(e_k(z))\prod_{j=1}^{k} e_j(z)\,dz}{\Lambda_k(e_k(z))}.$$

But $$\Lambda_k(e_k(z)) = \prod_{j=0}^{k} \log_j(e_k(z)) = \prod_{j=0}^{k} e_{k-j}(z) = \prod_{j=0}^{k} e_j(z)$$

so finally

$$l_k(x) = \exp \int_{\log_k \zeta}^{x} \delta(e_k(z))/z\, dz$$

which shows that $l_k(x)$ is S. V. Hence Theorem 1 of [1] yields

$$I(n) = \int_0^{\log_k n} \frac{dy}{y^\rho l_k(y)} + 0(1)$$

$$= (\log_k n)^{1-\rho} \int_0^1 \frac{dz}{z^\rho l_k(z \log_k n)} + 0(1)$$

$$\sim \frac{(\log_k n)^{1-\rho}}{(1-\rho)\, l_k(\log_k n)} = \frac{(\log_k n)^{1-\rho}}{(1-\rho)\, L_k(n)}.$$

Thus we finally obtain

$$\sum_{m=1}^{n} t_m \sim A \frac{(\log_k n)^{1-\rho}}{(1-\rho)\, L_k(n)}.$$

Similarly, if $\rho > 1$ we obtain

$$\sum_{m=n}^{\infty} t_m \sim \frac{A}{(\rho-1)(\log_k n)^{\rho-1} L_k(n)}$$

for the rate of convergence of $\sum_{n=1}^{\infty} t_n$.

(ii) *Gauss' Test.* If

$$t_n/t_{n+1} = 1 + \sum_{j=0}^{k-1} \frac{1}{\Lambda_j(n)} + \rho/\Lambda_k(n) + \varepsilon(n)/\Lambda_k(n)$$

where $\varepsilon(n) \to 0$ $(n \to \infty)$ then it is clear that $D_k(\mathcal{J}, n) \to \rho$. If $\rho = 1$ and

$$\varepsilon(n) = 0[(\log_k n)^{-\lambda}].$$

for some $\lambda > 0$, then

$$D_{k+1}(\mathcal{J}, n) = 0[\Lambda_k(n)(\log_k n)^{\lambda}]$$

$$= 0[\log(\log_k n)/(\log_k n)^{\lambda}] \to 0$$

so that $\sum t_n$ diverges. The case $k = 0$ is Gauss' test [9, p. 288].

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