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ON THE CANONICAL FORMALISM WITH THE DERIVATIVES OF HIGHER ORDER

Đorđe Mušicki

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Abstract. After a survey of the generalized Hamilton's equations with the derivatives of higher order, the corresponding theory of canonical transformations is given in the most general case, including also the results of earlier authors: the relations with the generating function, infinitesimal transformations, the Hamilton-Jacobi method, Lagrange and Poisson brackets, as well as integral invariants of the first and higher order and the corresponding Liouville's theorem.

1. Introduction

Some thirty years ago, F. Bopp and B. Podolski attempted a generalization of electrodynamics, based on the Lagrangian with the derivatives of the second order. Then the interest has been raised for the generalization of the common Hamiltonian canonical formalism to the case when the derivatives of arbitrary order appear in the Lagrangian. Several research phases and groups of scientific papers dealing with the above problem are to be distinguished:

a) The papers based on the calculus of variations (M. Ostrogradski 1850, Th. de Donder 1929)

b) The attempts to generalize the electrodynamics (F. Bopp 1940, B. Podolski, C. Kikuchi and P. Schwed 1942—1948).

c) The generalization of the Hamiltonian formalism (T. Chang 1946, J. de Wet 1948, M. Borneas 1959, J. Koestler and J. Smith 1965, K. Thielheim 1967, L. Coelho de Souza, L. and P. Rodrigues 1969).

d) The quantization on the basis of this formalism (D. Montgomery 1946, J. de Wet 1948, A. Pays and G. Uhlenbeck 1950, G. Hayes 1968, M. Borneas 1972).

Let us consider here only those papers which pertain to the mechanics of particles. M. Ostrogradski [1] has shown that the Euler-Lagrange equations of the calculus of variations can always be substituted by the equivalent differential equations of the first order. Starting from his own invariant theory of the calculus of variations, Th. de Donder [2] obtained the equations of the extremals in the Hamiltonian form as well as the corresponding Hamilton-Jacobi equation.

T. Chang [3] found indirectly the Hamilton's equations for this case, but without explicit definitions of the canonical variables, which must satisfy a system of differential equations. M. Borneas [4] was the first to formulate explicitly the corresponding Hamiltonian and the generalized momenta, transforming the Lagrange's equations into such a chape from which the law of energy conservation follows.

The corresponding generalized Hamilton's equations were also obtained by J. Koestler and J. Smith [5], independently of Ostrogradski and de Donder. They obtained it in a manner similar to that used in the analytical mechanics and formulated the corresponding Poisson and Lagrange brackets. After accepting certain remarks of the two authors [6], Ligia and P. Rodrigues [7] have formulated also the corresponding canonical transformations. They corrected the definition of the Poisson bracket given by the mentioned authors so that this is always invariant, and obtained the corresponding Hamilton-Jacobi equation on the basis of the canonical transformations.

2. Several formulae from the calculus of functionals

Two basic operations may be introduced for the functionals, which will be used further in this paper, and for which the corresponding calculus of functionals was worked out by V. Volterra [8]. The derivative of the functional $F=F[\varphi_i(x_\alpha, \tau)]$ with respect to $\varphi_i(x)$ in the point M is defined as

$$(1)\qquad \left.\begin{aligned} \frac{\delta F}{\delta \varphi_i(x)} &= \lim_{\Delta\sigma\to 0} \frac{F[\ldots, \varphi_i+\Delta\varphi_i, \ldots]-F[\ldots, \varphi_i, \ldots]}{\Delta\sigma} \\ \Delta\sigma &\equiv \int \Delta\varphi_i(x)\, dV, \end{aligned}\right\}$$

where $\Delta\varphi_i$ is the variation of the function $\varphi_i(x_\alpha, \tau)$ in the vicinity ΔV of the point M, and that represents the generalization of the notion of the partial derivative. The differential of this functional is then defined by the formula

$$(2)\qquad dF=\int \sum_i \frac{\delta F}{\delta \varphi_i(x)}\, d\varphi_i(x)\, dV,$$

where $d\varphi_i(x)$ are the increments of the functions $\varphi_i(x_\alpha, \tau)$ in the point M, and this represent the generalization of the notion of total differential.

Consider the functionals in the form of an integral

$$F=\int \mathcal{F}(\varphi_i;\ \varphi_{i,\alpha_1};\ \varphi_{i,\alpha_1\alpha_2};\ \ldots;\ x_\alpha)\, dV,$$

where $\varphi_{i,\alpha}=\partial\varphi_i/\partial x_\alpha$ and so on. With the above definition, the development of the integrand into Taylor's series and the transformation of the obtained integrals using the integration by parts, yield

$$\left.\begin{aligned}\frac{\delta F}{\delta\varphi_i(x)}=\frac{\partial\mathcal{F}}{\partial\varphi_i}-\sum_{\alpha_1}\frac{d}{dx_{\alpha_1}}\frac{\partial\mathcal{F}}{\partial\varphi_{i,\alpha_1}}+\sum_{\alpha_1}\sum_{\alpha_2}\frac{d^2}{dx_{\alpha_1}dx_{\alpha_2}}\frac{\partial\mathcal{F}}{\partial\varphi_{i,\alpha_1\alpha_2}}-\\ -\cdots+(-1)^s\sum_{\alpha_1}\cdots\sum_{\alpha_s}\frac{d^s}{dx_{\alpha_1}\cdots dx_{\alpha_s}}\frac{\partial\mathcal{F}}{\partial\varphi_{i,\alpha_1\cdots\alpha_s}},\end{aligned}\right\} \tag{3}$$

where s is the order of the highest derivative, and this for the case $\varphi_i=y_i^{(m)}(x)$ is reduced to

$$\frac{\delta F}{\delta y_i^{(m)}}=\frac{\partial\mathcal{F}}{\partial y_i^{(m)}}-\frac{d}{dx}\frac{\delta\mathcal{F}}{\partial y_i^{(m+1)}}+\cdots+(-1)^{s-m}\frac{d^{s-m}}{dx^{s-m}}\frac{\partial\mathcal{F}}{\partial y_i^{(s)}}. \tag{4}$$

For the functionals in the form of a function

$$F=f(x')=\int K(x,x')\,\varphi(x)\,dV,$$

the general definition of the functional derivative gives

$$\frac{\delta f(x')}{\delta\varphi(x)}=K(x,x'), \tag{5}$$

and in the special case when $f(x)=\varphi(x)$, on the basis of the definition of Dirac's function

$$\frac{\delta f(x')}{\delta f(x)}=\delta(x-x'). \tag{6}$$

3. Generalized Hamilton's equations

Let us consider now a system of particles with r degrees of freedom, and assume the possibility of describing it by certain Lagrangian of the form

$$L=L(q_k,\dot{q}_k,\ddot{q}_k,\ldots,q_k^{(s)},t). \tag{7}$$

The corresponding Lagrange's equations, which are equivalent to the Hamilton's principle, have in this case the following from

$$\frac{\partial L}{\partial q_k}-\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_k}+\frac{d^2}{dt^2}\frac{\partial L}{\partial\ddot{q}_k}-\cdots+(-1)^s\frac{d^s}{dt^s}\frac{\partial L}{\partial q_k^{(s)}}=0,$$

or more concisely

$$\frac{\delta W}{\delta q_k}=0, \qquad W\equiv\int_{t_0}^{t_1} L\,dt. \tag{8}$$

For such systems J. Koestler and J. Smith [5] obtained the corresponding canonical equations introducing the generalized momenta

$$p_{k/m}=\frac{\delta W}{\delta q_k^{(m)}}=\sum_{j=0}^{s-m}(-1)^j\frac{d^j}{dt^j}\frac{\partial L}{\partial q_k^{(j+m)}}. \tag{9}$$

In this way they substituted the Lagrange's equations by the equivalent system of equations

$$\left.\begin{array}{l} \dot p_{k/m}=-\dfrac{\partial H}{\partial q_k^{(m-1)}}, \quad \dot q_k^{(m-1)}=\dfrac{\partial H}{\partial p_{k/m}}, \\ (k=1, 2, \ldots, r; \quad m=1, 2, \ldots, s) \end{array}\right\} \tag{10}$$

where the Hamiltonian is

$$H(q_k^{(m-1)}, p_{k/m}, t)=\sum_{k=1}^{r}\sum_{m=1}^{s} p_{k/m}\,q_k^{(m)}-L. \tag{11}$$

These are the generalized Hamilton's equations for this case, and they represent a system of $2\,rs$ differential equations of the first order with unknown functions

$$q_k,\ \dot q_k, \ldots, q_k^{(s-1)};\quad p_{k/1},\ p_{k/2}, \ldots, p_{k/s},$$

which here take the role of the canonical variables. This Hamil onian could be formed by eliminating only the derivatives of the highest oıder, i. e. by solving the last equations (9) for $m=s$ with respect to the derivatives of the highest order $q_k^{(s)}$. Inserting these functions into the expression (11), one effectuates the transition from Lagrange's to Hamilton's formalism.

In certain cases these systems can be reduced to the standard systems, namely when

$$L(q_k, \dot q_k, \ldots, q_k^{(s)}, t)=L_0(q_k, \dot q_k, t)+\frac{d}{dt}f(q_k, \dot q_k, \ldots, q_k^{(s-1)}, t). \tag{12}$$

Then it will be

$$\frac{\partial L}{\partial q_k^{(s)}}=\frac{\partial}{\partial q_k^{(s)}}\left\{\sum_k\sum_m\frac{\partial f}{\partial q_k^{(m-1)}}\,q_k^{(m)}\right\}=\frac{\partial f}{\partial q_k^{(s-1)}},$$

from which it follows

$$\Delta\equiv\left|\frac{\partial^2 L}{\partial q_k^{(s)}\,\partial q_l^{(s)}}\right|=0. \tag{13}$$

Consequently, these exceptional systems are degenerated in the sense of Dirac [9, 10].

4. Formulation of the canonical transformations

As usually, the canonical transformations are defined [7] as such transformations of canonical variables

(14) $$Q_k^{(m-1)}=F_{km}(q_k^{(m-1)},\ p_{k/m},t),\quad P_{k/m}=G_{km}(q_k^{(m-1)},\ p_{k/m},t),$$

which maintain the shape of the generalized Hamilton's equations invariant and which may be solved with respect to the old canonical variables. The Hamilton's equations being equivalent to the Hamilton's principle in the form

$$\delta\int_{t_0}^{t_1}(\sum_k\sum_m p_{k/m}\,q_k^{(m)}-H)\,dt=0,$$

the necessary and sufficient condition for the transformation (14) to be canonical is

(15) $$\sum_k\sum_m p_{k/m}\,dq_k^{(m-1)}-H\,dt=c\,(\sum_k\sum_m P_{k/m}\,dQ_k^{(m-1)}-K\,dt)+d\,G,$$

where K is the new Hamiltonian, and G is the corresponding generating function. Due to the factor c, a wider class of canonical transformations is included than usually.

For the generating function of the type $G_1(q_k^{(m-1)},\ Q_k^{(m-1)},\ t)$ if it is explicitly written and the corresponding coefficients compared, one obtains

(16) $$\left.\begin{aligned} p_{k/m}=\frac{\partial G_1}{\partial q_k^{(m-1)}},\qquad c\,P_{k/m}=-\frac{\partial G_1}{\partial Q_k^{(m-1)}}\\ cK=H+\frac{\partial G_1}{\partial t}\end{aligned}\right\}$$

If the condition (15) is transformed so that the independent variables are $q_k^{(m-1)}$, $P_{k/m}$ and t, and the generating function

$$G_2(q_k^{(m-1)},\ P_{k/m},t)=G_1+c\sum_k\sum_m P_{k/m}\,Q_k^{(m-1)},$$

is introduced, using a similar procedure it is found that

(17) $$\left.\begin{aligned} p_{k/m}=\frac{\partial G_2}{\partial q_k^{(m-1)}},\qquad c\,Q_k^{(m-1)}=\frac{\partial G_2}{\partial P_{k/m}}\\ cK=H+\frac{\partial G_2}{\partial t}\end{aligned}\right\}$$

Solving the first system of equations with respect to $Q_k^{(m-1)}$ and $P_{k/m}$ respectively, and introducing them into the second system, the new canonical variables will be obtained as the functions of the old ones.

5. Infinitesimal canonical transformations

Let us define these transformations by the relations

(18) $$Q_k^{(m-1)} = q_k^{(m-1)} + \delta q_k^{(m-1)}, \quad P_{k/m} = p_{k/m} + \delta p_{k/m}.$$

Since the generating function

$$G_2^0 (q_k^{(m-1)}, P_{k/m}) = \sum_k \sum_m q_k^{(m-1)} P_{k/m},$$

gives an identical transformation, for the generating function of the considered canonical transformation we may take

(19) $$G_2 (q_k^{(m-1)}, P_{k/m}, t) = \sum_k \sum_m q_k^{(m-1)} P_{k/m} + \varepsilon G' (q_k^{(m-1)}, P_{k/m}, t),$$

where ε is a small parameter. According to (17) one then obtains

(20) $$\delta q_k^{(m-1)} = \varepsilon \frac{\partial G'}{\partial p_{k/m}}, \quad \delta p_{k/m} = -\varepsilon \frac{\partial G'}{\partial q_k^{(m-1)}}.$$

In the special case $G' = H$, $\varepsilon = dt$ these relations, with the aid of the generalized Hamilton's equations, give

(21) $$\delta q_k^{(m-1)} = d q_k^{(m-1)}, \quad \delta p_{k/m} = d p_{km},$$

i. e., even in this case the evolution of the state of the system in the course of time may be considered as a sequence of successive infinitesimal canonical transformations.

For the variation of any function of canonical variables in the sense

$$\delta F = F (Q_k^{(m-1)}, P_{k/m}, t) - F (q_k^{(m-1)}, p_{k/m}, t),$$

developing the first term into Taylor's series and substituting $\delta q_k^{(m-1)}$ and $\delta p_{k/m}$ according to (20), we find

(22) $$\delta F = \varepsilon [F, G'],$$

where the symbol [ ] represents the generalized Poisson bracket

(23) $$[u, v] = \sum_k \sum_m \left(\frac{\partial u}{\partial q_k^{(m-1)}} \frac{\partial v}{\partial p_{k/m}} - \frac{\partial u}{\partial p_{k/m}} \frac{\partial v}{\partial q_k^{(m-1)}} \right).$$

Using this notion, one can formulate also the condition for certain quantity to be a constant of motion, as it is in the common case

$$\frac{dF}{dt} = [F, H] + \frac{\partial F}{\partial t} = 0.$$

6. Hamilton-Jacobi method

Taking such a canonical transformation for which the new Hamiltonian is identically zero, the generalized Hamilton's equations in the new variables give

$$P_{k/m} = \alpha_{km} = \text{const}, \qquad Q_k^{(m-1)} = \beta_{km} = \text{const}. \tag{24}$$

If the generating function of the second type is taken, and all the generalized momenta $p_{k/m}$ in the Hamiltonian $H(q_k^{(m-1)}, p_{k/m}, t)$, according to the first relation (17), are substituted by $\partial S / \partial q_k^{(m-1)}$, the last relation yields

$$\frac{\partial S}{\partial t} + H\left(q_k^{(m-1)}, \frac{\partial S}{\partial q_k^{(m-1)}}, t\right) = 0. \tag{25}$$

This is the corresponding Hamilton-Jacobi equation [7]. If, by using a similar procedure as that in the analytical mechanics, we find one its complete integral

$$S = S(q_k^{(m-1)}, \alpha_{km}, t),$$

the solutions of the generalized Hamilton's equations may be obtained with the aid of the system of corresponding equations (17)

$$\frac{\partial S(q_k^{(m-1)}, \alpha_{km}, t)}{\partial q_k^{(m-1)}} = p_{k/m}, \qquad \frac{\partial S(q_k^{(m-1)}, \alpha_{km}, t)}{\partial \alpha_{km}} = c\beta_{km}. \tag{26}$$

By algebraic solution of these equations with respect to $q_k^{(m-1)}$ and $p_{k/m}$ one can evaluate all canonical variables as the functions of time and integration constants.

7. Necessary and sufficient condition in the form of brackets

The condition (15) for the canonical transformation may be expressed also by primary variables only, substituting $dQ_k^{(m-1)}$ according to (14) by a corresponding total differential. Hence, one obtains

$$\sum_k \sum_m \left(p_{k/m} - c \sum_{k'} \sum_{m'} P_{k'/m'} \frac{\partial Q_{k'}^{(m'-1)}}{\partial q_k^{(m-1)}}\right) dq_k^{(m-1)} + \sum_k \sum_m \left(-c \sum_{k'} \sum_{m'} P_{k'/m'} \times \right.$$

$$\left. \times \frac{\partial Q_{k'}^{(m'-1)}}{\partial p_{k/m}}\right) dp_{k/m} + \left(cK - H - c \sum_{k'} \sum_{m'} P_{k'/m'} \frac{\partial Q_{k'}^{(m'-1)}}{\partial t}\right) dt = dG.$$

Using the conditions for which the expression on the left side is a total differential, one can write them in a more concise form by introducing the generalized Lagrange bracket

$$\{u, v\} = \sum_k \sum_m \left(\frac{\partial q_k^{(m-1)}}{\partial u} \frac{\partial p_{k/m}}{\partial v} - \frac{\partial q_k^{(m-1)}}{\partial v} \frac{\partial p_{k/m}}{\partial u}\right), \tag{27}$$

so that these conditions are reduced to

$$(28)\qquad \left.\begin{aligned} \{q_k^{(m-1)},\ q_l^{n-1}\}_{Q,P}=0,\quad \{p_{k|m},\ p_{l|n}\}_{Q,P}=0,\\ \{q_k^{(m-1)},\ p_{l|n}\}_{Q,P}=\frac{1}{c}\,\delta_{kl}\,\delta_{mn}\end{aligned}\right\}$$

Let us mention that this bracket represents a modification of the definition given by earlier authors [5], by analogy with the Poisson bracket.

These Lagrange and Poisson brackets are related by

$$\sum_{l=1}^{2rs}\{u_l,\ u_i\}\,[u_l,\ u_j]=\delta_{ij},$$

where u_l are the arbitrary functions of the canonical variables. On the basis of this relation and the conditions (28) may be obtained the equivalent conditions in the form of the generalized Poisson brackets also

$$(29)\qquad \left.\begin{aligned} [q_k^{(m-1)},\ q_l^{(n-1)}]_{Q,P}=0,\quad [p_{k|m},\ p_{l|n}]_{Q,P}=0,\\ [q_k^{(m-1)},\ p_{l|n}]_{Q,P}=c\,\delta_{kl}\,\delta_{mn}\end{aligned}\right\}$$

8. Invariance of the Lagrange and Poisson brackets

If we form the generalized Lagrange bracket (27) in the new canonical variables and if we transform all partial de ivatives to the old variables, we will find

$$\{u,\ v\}_{Q,P}=\sum_l\sum_n\sum_{l'}\sum_{n'}\left(\frac{\partial q_l^{(n-1)}}{\partial u}\,\frac{\partial q_{l'}^{(n'-1)}}{\partial v}\,\{q_l^{(n-1)},\ q_{l'}^{(n'-1)}\}_{Q,P}+\right.$$

$$+\frac{\partial q_l^{(n-1)}}{\partial u}\,\frac{\partial p_{l'|n'}}{\partial v}\,\{q_l^{(n-1)},\ p_{l'|n'}\}_{Q,P}+\frac{\partial p_{l|n}}{\partial u}\,\frac{\partial q_{l'}^{(n'-1)}}{\partial v}\,\{p_{l|n},\ q_{l'}^{(n'-1)}\}_{Q,P}+$$

$$\left.+\frac{\partial p_{l|n}}{\partial u}\,\frac{\partial p_{l'|n'}}{\partial v}\,\{p_{l|n},\ p_{l'|n'}\}_{Q,P}\right).$$

In this way this Lagrange bracket is reduced to the fundamental one and with the aid of (28) one obtains

$$(30)\qquad \{u,\ v\}_{Q,P}=\frac{1}{c}\,\{u,\ v\}_{q,p}.$$

In the same manner, transforming the generalized Poisson bracket one finds

$$(31)\qquad [u,\ v]_{Q,P}=c\,[u,\ v]_{q,p}.$$

Therefore, the generalized Lagrange and Poisson brackets remain invariant in the canonical transformations up to the multiplier $1/c$ i. e. c.

9. Generalized de Donder's relation

In order to represent geometrically the states of the system in this case, let us introduce the generalized phase space as the Euclidean space with $2rs$ dimensions, the points of which are the sets

$$\{q_k, \dot{q}_k, \ldots, q_k^{(s-1)}, p_{k/1}, p_{k/2}, \ldots, p_{k/s}\}.$$

In thus defined space one can introduce the corresponding integral invariants by generalizing one of de Donder's relations in the following manner.

Let us consider the case when the canonical variables are dependent not only on time, but on some parameter α also

$$q_k^{(m-1)} = q_k^{(m-1)}(t, \alpha), \quad p_{k/m} = p_{k/m}(t, \alpha) \tag{32}$$

If we denote by δ the variation which comes from the change of this parameter, and if we put $q_{r+1} = t$, $p_{r+1} = -H$, we can introduce the quantity

$$j = \sum_{k'=1}^{r+1} \sum_{m=1}^{s} \frac{\delta W}{\delta q_{k'}^{(m)}} \delta q_{k'}^{(m-1)} = \sum_{k=1}^{r} \sum_{m=1}^{s} \frac{\delta W}{\delta q_k^{(m)}} \delta q_k^{(m-1)} - H \delta t. \tag{33}$$

Starting from the identity, which results from the definition of the functional derivative

$$\frac{\delta W}{\delta q_k^{(m-1)}} = \frac{\partial L}{\partial q_k^{(m-1)}} - \frac{d}{dt} \frac{\delta W}{\delta q_k^{(m)}},$$

and generalizing the relation between the variation of a derivative and the derivative of the variation correspondent to the derivatives of arbitrary order

$$\delta q_k^{(m)} = \frac{d}{dt} \delta q_k^{(m-1)} - q_k^{(m)} \frac{d}{dt} \delta t,$$

after multiplying this identity by $\delta q_k^{(m-1)}$ and summation with respect to indices, one obtains

$$\frac{dj}{dt} = \delta L - \sum_{k'} \frac{\delta W}{\delta q_{k'}} \delta q_{k'} + \sum_{k'} \sum_{m} \frac{\delta W}{\delta q_{k'}^{(m)}} q_{k'}^{(m)} \frac{d}{dt} \delta t. \tag{34}$$

This relation represents the generalization of the central Lagrange's equation in common analytical mechanics, and in the special case when $\delta t = 0$ is reduced to the de Donder's relation (Ref. 2, p. 98).

10. Lagrange's equations and integral invariants of the first order

If the Lagrange's equations (2) are satisfied, the second term on the right side of (34) vanishes. By integrating the above relation along the curve (32), i. e. with respect to the parameter α, not varying the time within the limits but only the limits $t_1(\alpha)$ and $t_2(\alpha)$, one obtains

$$\frac{d}{dt} \int_1^2 j = \frac{d}{dt} \int_1^2 \sum_{k'} \sum_{m} \frac{\delta W}{\delta q_{k'}^{(m)}} \delta q_{k'}^{(m-1)} + \int_1^2 \delta L.$$

If this curve is closed, the last integral at the right side is zero, and hence the quantity

$$\mathcal{J} = \oint \sum_{k'} \sum_{m} \frac{\delta W}{\delta q_{k'}^{(m)}} \delta q_{k'}^{(m-1)} = \oint \Big(\sum_{k} \sum_{m} p_{k/m} \delta q_{k}^{(m-1)} - H \delta t\Big), \tag{35}$$

taken along any curve in our space remains costant during the time.

This integral represents the generalization of the Poincaré-Cartan integral invariant to our case, its geometiical sense being as follows. If in the enlarged phase space (Fig. 1) we imagine any closed curve L_0, and through each point

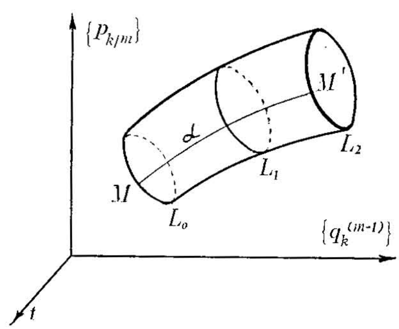

Fig. 1

of this curve the corresponding trajectory of state is drawn, then the integral (35) has the same value along all curves L_0, L_1, L_2 .... which lie on the surface formed by these trajectories. It should be mentioned that the same results can be obtained starting from the corresponding general variation of action

$$\delta W = \Big| \Big(\sum_{k} \sum_{m} p_{k/m} \delta q_{k}^{(m-1)} - H \delta t\Big) \Big|_0^1 .$$

In order to investigate the behavior of this integral in the canonical transformations, we shall integrate the necessary and sufficient condition (15), where d is substituted by δ, along any closed curve

$$\oint_L \Big(\sum_{k} \sum_{m} p_{k/m} \delta q_{k}^{(m-1)} - H \delta t\Big) = c \oint_{\bar{L}} \Big(\sum_{k} \sum_{m} P_{k/m} \delta Q_{k}^{(m-1)} - K \delta t\Big) + \oint_L \delta G,$$

where $\bar{L}$ represents the corresponding curve in the phase space of new canonical variables. Since $\oint \delta G = 0$, one obtains

$$\bar{\mathcal{J}} = \frac{1}{c} \mathcal{J}, \tag{36}$$

i. e. the generalized Poincaré-Cartan integral is invariant with respect to the canonical transformations up to the multiplier $1/c$.

In the special case $\delta t = 0$, when the time does not vary, the integral (35) is reduced to

(37) $$\mathcal{J}_1 = \oint \sum_k \sum_m p_{k/m} \delta q_k^{(m-1)},$$

and that is the generalization of the Poincaré's integral invariant.

11. Absolute integral invariants of higher order

The last integral may be transformed into a surface integral by using Stokes' theorem

(38) $$\mathcal{J}_1 = \oint_L \sum_k \sum_m p_{k/m} \delta q_k^{(m-1)} = \int_S \sum_k \sum_m \delta q_k^{(m-1)} \delta p_{k/m}.$$

In analogy with Poincaré's integral invariants of higher order in the analytical mechanics [11], let us form the following set of integrals, where the sum in the integrand is to be taken only over the different indices

(39) $$\left.\begin{aligned} \mathcal{J}_1 &= \int_S \sum_k \sum_m \delta q_k^{(m-1)} \delta p_{k/m}, \\ \mathcal{J}_2 &= \int_S \int_S \sum_k \sum_{k'} \sum_m \sum_{m'} \delta q_k^{(m-1)} \delta q_{k'}^{(m'-1)} \delta p_{k/m} \delta p_{k'/m'} \\ &- \\ &- \\ \mathcal{J}_{rs} &= \int\int\cdots\int \delta q_1 \delta \dot{q}_1 \ldots \delta q_1^{(s-1)} \ldots . \delta q_r \delta \dot{q}_r \ldots \delta q_r^{(s-1)} \times \\ &\times \delta p_{1/1} \delta p_{1/2} \ldots \delta p_{1/s} \ldots . . \delta p_{r/1} \delta p_{r/2} \ldots \delta p_{r/s}. \end{aligned}\right\}$$

The last of these integrals may be interpreted as a volume in our phase space.

Introducing the Gauss' parameters as the integration variables, the element of each of these integrals may be written as

$$d\mathcal{J}_f = \frac{\partial (q_{k_1}^{(m_1-1)}, \ldots, q_{k_f}^{(m_f-1)}, p_{k_1/m_1}, \ldots, p_{k_f/m_f})}{\partial (u_1, u_2, \ldots, u_{2f})} du_1 \, du_2 \ldots . du_{2f}.$$

Let us develop the Jacobian of this transformation with respect to the minors of the second order, generalizing the corresponding relation from the analytical mechanics [12],

$$\sum_{k_1} \cdots \sum_{k_f} \sum_{m_1} \cdots \sum_{m_f} \frac{\partial (q_{k_1}^{(m_1-1)}, \ldots, q_{k_f}^{(m_f-1)}, p_{k_1/m_1}, \ldots, p_{k_f/m_f})}{\partial (u_1, u_2, \ldots, u_{2f})} = \frac{1}{2^f} \sum \pm \{u_1, u_2\} \times$$
$$\times \{u_3, u_4\} \cdots \{u_{\nu_{2f-1}}, u_{\nu_{2f}}\}.$$

Here the summation is extended over all the permutatious $\{\nu_1, \nu_2, \ldots, \nu_{2f}\}$ denoted by + or − depending on whether the corresponding permutation is odd or even. In this way, on the basis of (30) one obtains

$$\overline{\mathcal{J}}_f = \frac{1}{c^f} \mathcal{J}_f \quad (1 \leqslant f \leqslant rs) \tag{40}$$

Therefore, all generalized Poincaré's integrals of higher order remain invariant under canonical transformations up to the multiplier $1/c^f$.

12. Generalized Liouville's theorem

As a consequence of the invariance of the last integral (39) even in this case follows the corresponding Liouville's theorem. Indeed, for $f = rs$ and $c = 1$ one has

$$\overline{\mathcal{J}}_{rs} = \mathcal{J}_{rs} \quad (c = 1)$$

and this integral represents the volume $\Delta\Gamma$ in the phase space, i. e.

$$\overline{\Delta\Gamma} = \Delta\Gamma. \tag{41}$$

On the other hand, it is shown that in this case also the development of the state of the system in the course of time may be represented as a sequence of successive infinitesimal canonical transformations.

Hence, as in the analytical mechanics, the result (41) can be interpreted in the following way. If $\Delta N = \rho_1 \Delta\Gamma_1$ of the representative points in the phase space in the moment t_1 occupies a volume $\Delta\Gamma_1 = \Delta\Gamma$, then in some later moment t_2 they will occupy another part of the phase space of the same volume $\Delta\Gamma_2 = \overline{\Delta\Gamma}$. Since $\Delta N = \rho_1 \Delta\Gamma_1 = \rho_2 \Delta\Gamma_2$, it follows

$$\rho \equiv \rho_1 = \rho_2 = \text{const} \tag{42}$$

and this may be written in the form

$$\frac{d\rho}{dt} = [\rho, H] + \frac{\partial \rho}{\partial t} = 0. \tag{43}$$

Accordingly, the density of the representative points along their trajectories in the phase space remains constant in the course of time, i. e. the Liouville's theorem remains valid in this generalized case also. Therefore, one can conclude that the common statistical physics can be extended also to the case when the Hamiltonian of the system depends on the derivatives of arbitrary order.

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Faculty of Nat. and Math. Sciences
and Institute of Physics,
Beograd, Yugoslavia