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A PROOF AND AN EXTENSION OF A THEOREM OF G. BIRKHOFF

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In [1, Theorem 2. p. 182] Birkhoff proved the following theorem utilizing the axiom of choice:

Theorem. Let E be the universal semilattice of the partially ordered set P. Then E fulfils the descending chain condition iff P does.

In the paper [2] this theorem is proved by utilizing the Tichonof's theorem. Here, we give a proof by utilizing the Köning's lemma. Previously we introduce some definitions and notions.

Let P be a partially ordered set. The lower end of P is any subset L of P, for which $p \in L$ and $q \leqslant p$ are fulfilled, then $q \in L$. The family $\mathscr{L}(P)$ of all lower ends of P i.e. of all finitely generated lower ends is a semilattice with respect to the inclusion and the set theoretic union as a join.

Definition. E is a universal semilattice of P if for every semilattice F and every isotone mapping $f: P \to F$ can be in a unique way lengthened to a homomorphism $\varphi: E \to F$.

In the paper [3] was given the following assertion: E is a universal semilattice of P if and only if every element $x \in E$ has a unique representation as $\sup C(x)$, $C(x)$ being a finite antichain $C(x) \subset P$.

So the semilattice $\mathscr{L}(P)$ can be considered as an abstract extension E of P [1, 3].

2. For every $x \in E$ we have $x = \sup C(x)$ where $C(x)$ is a finite antichain from P.

Lemma. *Let $x, y \in E$. Then*

$$x < y \Rightarrow (\forall p \in C(x))\,(\exists q \in C(y))\,(p \leqslant q).$$

Proof. Let us start from the contrary i.e.

$$x < y \text{ and } (\exists p \in C(x))\,(\forall q \in C(y))\,(p \not\leqslant q).$$

Then we have two possibilities form that p from $C(x)$:

(a) $(\forall q \in C(y))\ (p > q)$ and

(b) p is comparable to no q from $C(y)$.

If (a) is fulfilled, then $y = \sup C(y) \leqslant p$, but as $p \leqslant \sup C(x) = x < y \leqslant p$ which is a contradiction.

If (b) is fulfilled, then because of $p \leqslant x < y$, we have $y = \sup C(y)$ and $y = \sup (C(y) \cup \{p\})$ which gives a second representation for the element y. Contradiction.

In that way the Lemma is proved.

Proof of theorem. As $E \subset P$, then, if E satisfies the condition of descending chains, this feature has also P too (trivially). Contrarily let us prove the following assertion:

If P satisfies the descending chain condition, so does E too.

Suppose the contrary, i. e. that P satisfies the descending chain condition and E does not satisfy it. Then in E there should be a strongly descending chain

$$(1) \qquad x_1 > x_2 > \cdots > x_n > \cdots$$

where $x_i \in E$ $(i = 1, 2, \ldots)$. As $x_i = \sup C(x_i)$, we can assume for some $s \in N$ and every $n \geqslant s$ $\bigcap\limits_{k \geqslant n} C(x_k) = \varnothing$.

Let us notice that the following set of formulas

$$(\forall p \in C(x_{i+1}))\ (\exists q \in C(x_i))\ (p \leqslant q)$$

which are true on the basis of the proved lemma, because $x_{i+1} < x_i$ $(i = 1, 2, \ldots)$. Then on the basis of lemma for each (sufficiently large) n there exists a finite chain of elements from P

$$p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n, \quad p_i \in C(x_i) \subset P, \qquad (i = 1, 2, \ldots, n).$$

As all sets $C(x_i)$ are finite, then also on the basis of König's lemma [4] exists the infinite sequence

$$(2) \qquad p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n \geqslant \cdots$$

of elements from P, where $p_i \in C(x_i)$ $(i = 1, 2, \ldots)$.

Among all elements p_i in (2) there must be infinitely many different ones and they would form an infinite descending chain in P, a contradicton. As a matter of fact, if the set $\{p_n\}_{n \in N}$ were finite, then for some $s \in N$ one would have $p_s = p_{s+1} = \cdots$, thus $\bigcap\limits_{n \geqslant s} C(x) \neq \varnothing$, contradicting the assumption.

By this, the theorem is proved.

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