PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 23 (37) 1978, pp. 117—127

ON SOLVING A SYSTEM OF BALANCED FUNCTIONAL EQUATIONS ON QUASIGROUPS I

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(Communicated January 7, 1977)

In the first part of this work we consider a special case of a system of balanced functional equations on quasigroups. A system is irreducible, general and of extended equality type. The results we obtain are the partial generalisation of those of B. Alimpić, J. Ušan, V. D. Belousov and E. S. Livšic (see [2], [6], [4]).

In the second part, we shall free ourselves from irreducibility, and, in the third part, from generality and equality of all participating terms.

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Basic properties of n-quasigroups are given in [1]. We introduce some new symbols and define new notions, not strictly connected with n-quasigroups.

Let Q be n-ary operation on a nonempty set S and $M \subset \{1, \ldots, n\}$. Let Q_M be a retract obtained from Q by replacing variables from those arguments of Q whose ordinal numbers do not belong to M, with previously selected elements from S. For example, if $Q(x_1, \ldots, x_5)$ and $a_1, \ldots, a_5 \in S$ are given, then $Q_{\{1,3,4\}}(x, y, z) = Q(x, a_2, y, z, a_5)$

By (Q) we denote operation with a retract Q.

Let term t be given, where all the operation symbols occur only once in the term t and are interpreted as operations of some nonempty set S (in the sequel we do not distinguish operations from symbols of operations). To every retract R of such an operation corresponds the function f_R which depends on term t. If $R(t_1, \ldots, t_m)$ is a subterm of t then $f_R(k)$ is a set of variables of a term t_k $(k=1, \ldots, m)$. If $M = \{i_1, \ldots, i_r\}$ $(i_1 < \cdots < i_r \leqslant n)$ then $f_{R_M}(k) = f_R(i_k)$ $(k \leqslant r)$. For example if $t = A(x_1, B(x_2, C(x_3, x_4)), x_5)$ then $f_A(1) = \{x_1\}$, $f_A(2) = \{x_2, x_3, x_4\}$, $f_A(3) = \{x_5\}$, $f_{A_{\{1,3\}}}(1) = f_A(1) = \{x_1\}$, $f_{A_{\{1,3\}}}(2) = f_A(3) = \{x_5\}$ and so on.

Let S_t be the set of operations (operational symbols) and variables of the term t. To every operation Q corresponds that subterm TQ of t for which Q is the first (main) symbol. Term x corresponds to variable x of the term $t\,(Tx = x)$. Function T, from S_t to the set of subterms of the term t, defines an order relation $\leqslant_t$ of the set S_t, antiisomorphic to the relation of being a subterm ($Q \leqslant_t R$ iff TR is a subterm of TQ). $(S_t, \leqslant_t)$ is a (lower) semilattice.

For every operation A from S_t we define the partial function N_A:

$$N_A(B)=\begin{cases} C, \text{ if } TB=B(t_1,\ldots,t_m),\ TC=t_k \text{ for some } k\leqslant n,\ C\leqslant_t A \\ \text{not defined, if not } B<_t A \end{cases}$$

We call $N_A(B)$ successor of B in the direction of A. If t is as before then $N_{x_2}(A)=B$. $N_{x_2}(B)=x_2$, $N_{x_1}(B)$ is not defined and so on.

We also define a binary function arg for all $A, B\in S_t$

$$\arg(A,\ B)=\begin{cases} \{0\}, \text{ if not } A<_t B \\ \{k\},\ TA=A(t_1,\ldots,t_m) \text{ and } TB \text{ is a subterm of } t_k \end{cases}$$

In our example $\arg(B,\ A)=\{0\}$, $\arg(B,\ x_2)=\{1\}$, $\arg(B,\ C)=\{2\}$, $\arg(B,\ x_3)=\{2\}$ and so on.

Let formula $t_1=t_2$ be given, whose variables are exactly $x_1,\ldots,x_n$. If $a_1,\ldots,a_n\in S$ are given and $M\subset\{1,\ldots n\}$ then M-consequence of $t_1=t_2$ is the equality obtained from $t_1=t_2$ by replacing corresponding elements of S for the variables $x_i\,(i\notin M)$ (and replacing all the operations with the retracts of corresponding operations). For example formula $A_{\{2\}}B_{\{1\}}C_{\{3\}}x_3=D_{\{2\}}E(x_3,\ x_4)$ is the $\{3,\ 4\}$-consequence of the formula

$$A(x_1,\ B(C(x_2,\ x_1,\ x_3),\ x_2))=D(x_1,\ E(x_3,\ x_4),\ x_2).$$

Let $t_1=t_2$ be a given formula whose variables are exactly $x_1,\ldots,x_n$ and let A and B be retracts of operations occuring in t_1 and t_2 respectively.

Definition. $A\leftrightarrow B(t_1=t_2)$ iff there exists $M\subset\{1,\ldots,n\}$ such that $\{x_i\,|\,i\in M\}=\{y_1,\ldots,y_m\}=\{z_1,\ldots,z_m\}$ and such that

$$\prod_{Q<(A)} Q_{\arg(Q,(A))}\,A\Big(\prod_{(A)<Q<y_1} Q_{\arg(Q,y_1)}\,y_1,\ldots,\prod_{(A)<Q<y_m} Q_{\arg(Q,y_m)}\,y_m\Big)= \tag{1}$$

$$=\prod_{Q<(B)} Q_{\arg(Q,(B))}\,B\Big(\prod_{(B)<Q<z_1} Q_{\arg(}\quad z_1,\ldots,\prod_{(B)<Q<z_m} Q_{\arg(Q,z_m)}\,z_m\Big)$$

is M-consequence of $t_1=t_2$.

$\prod$ is composition operator like in $\prod\limits_{1\leqslant i\leqslant n} Q_i\,x=Q_1\ldots Q_n\,x$ and $<$ is one of $<_{t_1}$, $<_{t_2}$. If we define $\langle A,\ B\rangle=\prod\limits_{(A)<Q<(B)} Q_{\arg(Q,(B))}$ and $\langle\cdot,\ B\rangle=$ $=\prod\limits_{Q<(B)} Q_{\arg(Q,(B))}$ then (1) can be written as:

$$\langle\cdot,\ A\rangle A(\langle A,\ y_1\rangle y_1,\ldots,\langle A,\ y_m\rangle y_m)=\langle\cdot,\ B\rangle B(\langle B,\ z_1\rangle z_1,\ldots,\langle B,\ z_m\rangle z_m)$$

Let Γ be the conjunction (a system) of formulas of the form $t_i=t_j$ $(1\leqslant i\leqslant j\leqslant \bar{n})$.

Definition. Retracts A and B are $\leftrightarrow$-related iff there exists an equation $t_i=t_j$ such that $A\leftrightarrow B(t_i=t_j)$.

Definition. Relation $\sim$ is reflexive and transitive closure of the relation $\leftrightarrow$ in the set of all retracts of operations occuring in Γ.

$\sim$ is obviously an equivalence relation and if some retracts A and B belong to the same equivalence class of $\sim$, then they are diisotopic.

Definition. Retract A is locally irreducible iff for every $M \subset \{1, \ldots, n\}$, for which $\{x_i \mid i \in M\} \cap f_A(k)$ is a singleton for all $k = 1, \ldots, |A|$, there exists a retract B such that (1) is the M-consequence of some formula $t_i = t_j$ from Γ.

Definition. Retract A is irreducible iff all retracts from $A^{\sim}$ are locally irreducible.

Definition. System Γ is irreducible iff all operations from Γ are irreducible.

At the end of the introduction, we define a relation of partial equivalence.

Definition. Relation $\equiv$ on S is a partial equivalence iff following formulas hold:

(PR) $$\forall x \forall y (x \equiv y \Rightarrow x \equiv x)$$

(S) $$\forall x \forall y (x \equiv y \Rightarrow y \equiv x)$$

(T) $$\forall x \forall y \forall z (x \equiv y \wedge y \equiv z \Rightarrow x \equiv z).$$

It is easy to check that being a partial equivalence on S is the same as being an equivalence on some subset of S.

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Let Γ be an irreducible system of general balanced functional equations on quasigroups, of extended equality type. In another word Γ can be written in the form: $t_1 = \cdots = t_{\bar{n}}$, where $t_i (i = 1, \ldots, \bar{n})$ are such terms that every variable occurs exactly once in every term $t_i (i = 1, \ldots, \bar{n})$.

Irreducibility is defined in the introduction, while generality means that all symbols of operations from Γ are different one from another. The only assumption is that these symbols denote quasigroups of given arity.

Let S be any nonempty set and $+$ an abelian group on S. If A is an operational symbol from Γ (of arity k), then quasigroup A is defined as $A(x_1, \ldots, x_k) = x_1 + \cdots + x_k$. Every term $t_i (i = 1, \ldots, \bar{n})$ then becomes sum $x_1 + \cdots + x_n$, so this is a solution of Γ, hence Γ is consistent.

Let a solution of Γ (on a set S) be given. We interpret operational symbols from Γ as respective quasigroups from the given solution.

Lemma 1. *For every operation A from Γ,* card $A^{\sim} \geq \bar{n}$.

Proof: Trivial.

Lemma 2. *If A is an operation of arity greater than* 2, *then* card $A^{\sim} = \bar{n}$.

Proof: Let A be an operation (from Γ) of arity $k \geqslant 2$. Then, in some of $t_i \, (i=1, \ldots, \bar{n})$ for example t_1, occur at least two operations.

(1) Let these operations be B and C, $B<C$ and in some $t_r \, (r=2, \ldots, \bar{n})$ there is exactly one operation from $A^{\sim}$.

Then for some $x_{j_1} \in f_B(1), \ldots, x_{j_{p-1}} \in f_B(p-1)$, $x_{j_{p+1}} \in f_B(p+1), \ldots, x_{j_k} \in$ $\in f_B(k)$ and every $x_{m_1} \in f_C(1), \ldots, x_{m_k} \in f_C(k)$ (C occurs in p-th argument of B) and some $D \in A^{\sim}$ occuring in t_r, $B \leftrightarrow D \, (t_1 = t_r)$ is $\{j_1, \ldots, j_{p-1}, m_1, j_{p+1}, \ldots, j_k\}$ – and $\{j_1, \ldots, j_{p-1}, m_2, j_{p+1}, \ldots, j_k\}$-consequence of $t_1 = t_r$ so x_{m_1} and x_{m_2} are in the same argument of D. Then $C \leftrightarrow D \, (t_1 = t_r)$ cannot be $\{m_1, \ldots, m_k\}$ – consequence of $t_1 = t_r$. So (1) is impossible.

(2) If we choose $2\bar{n}$ operations such that exactly two of them occur in every term $t_i \, (i=1, \ldots, \bar{n})$, then, if each pair is incomparable then not all of them belong to $A^{\sim}$. It is so by definition of $\sim$.

(3) If we choose $2\bar{n}$ operations $A_1, \ldots, A_{\bar{n}}$, $B_1, \ldots, B_{\bar{n}}$ such that $A_i < B_i$ and A_i, B_i occur in $t_i \, (i=1, \ldots, \bar{n})$ and such that $A_i \leftrightarrow A_j$, $B_i \leftrightarrow B_j$ $(i, j = 1, \ldots, \bar{n};$ $i<j)$, then not all of them belong to $A^{\sim}$ because of definition of $\sim$.

(4) Let B, C be $\sim$-related operations from t_1, $B<C$ and D, E operations from some t_r such that $B \leftrightarrow E$, $C \leftrightarrow D$ and $D<E$.

Let $V_{BD} \cup V_{BE} \cup V_{CD} \cup V_{CE}$ be a set of $2k-1$ variables such that $V_{BD} \cup V_{BE}$ is the set of arguments of B but not of C, $V_{CD} \cup V_{CE}$ is the set of arguments of C, $V_{BD} \cup V_{CD}$ is the set of arguments of D but not of of E and $V_{BE} \cup V_{CE}$ is the set of arguments of E.

We have following scheme:

$$\begin{tikzcd}[column sep=small]
V_{CD} \arrow[dr, dash] & & V_{CE} \arrow[dl, dash] \\
& C \arrow[dd, dash] & \\
V_{BD} \arrow[dr, dash] & & V_{BE} \arrow[dl, dash] \\
& B &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
V_{BE} \arrow[dr, dash] & & V_{CE} \arrow[dl, dash] \\
& E \arrow[dd, dash] & \\
V_{BD} \arrow[dr, dash] & & V_{CD} \arrow[dl, dash] \\
& D &
\end{tikzcd}$$

It must be $\operatorname{card} V_{BE} = \operatorname{card} V_{CD}$ and $\operatorname{card} V_{CE} = \operatorname{card} V_{BD} + 1$.

If $V_{BD} \neq \varnothing$ then the choice of variables $V_{BD} \cup V_{BE}$ and some variable $x \in V_{CE}$ shows that B is locally reducible. From $x \in V_{CE}$ and $V_{BE} \notin \varnothing$ it follows that cannot be $B \leftrightarrow D \, (\varphi_{1r})$ and from $V_{BD} \neq \varnothing$ it follows that cannot be $B \leftrightarrow E \, (t_1 = t_r)$.

If $V_{BD} = \varnothing$ then $V_{CE} = \{y\}$ and we have:

$$\begin{tikzcd}[column sep=small]
V_{CD} \arrow[dr, dash] & & y \arrow[dl, dash] & \\
& C \arrow[dr, dash] & & V_{BE} \arrow[dl, dash] \\
& & B &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
V_{BE} \arrow[dr, dash] & & y \arrow[dl, dash] & \\
& E \arrow[dr, dash] & & V_{CD} \arrow[dl, dash] \\
& & D &
\end{tikzcd}$$

The choice of variables V_{BE} and some variable $x \in V_{CD}$ shows that B is locally reducible. From $x \in V_{CD}$ it follows that $B \leftrightarrow D \, (t_1 = t_r)$ cannot be true and from $x \neq y$ it follows that $B \leftrightarrow E \, (t_1 = t_r)$ cannot be true, which is impossible.

(5) Let B, C be $\sim$-related incomparable operations from t_1 and in some $t_r\,(r=2, \ldots, \bar{n})$ there is exactly one operation $D \in A^{\sim}$.

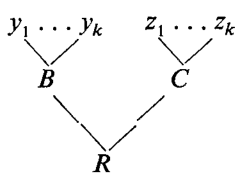
Then $B \leftrightarrow D\,(t_1=t_r)$ and $C \leftrightarrow D\,(t_1=t_r)$, so there are binary retracts R, $R_1, \ldots, R_k$ such that for some $y_1, \ldots, y_k, z_1, \ldots, z_k \in \{x_1, \ldots, x_n\}$ we have:

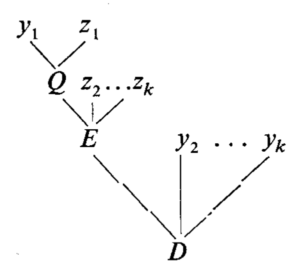
$$R(B(y_1, \ldots, y_k), C(z_1, \ldots, z_k)) = D(R_1(y_{1\pi}, z_{1\sigma}), \ldots, R_k(y_{k\pi}, z_{k\sigma}))$$

Where π and σ are permutations of $\{1, \ldots, k\}$. The choice of variables $y_{1\pi}, \ldots, y_{(k-1)\pi}, z_{k\sigma}$ contradicts the irreducibility of D.

(6) Let B, C be $\sim$-related incomparable operations from t_1 and D, E operations from some t_r such that $B \leftrightarrow D$, $C \leftrightarrow E$ and $D < E$.

Then for some $y_1, \ldots, y_k, z_1, \ldots, z_k \in \{x_1, \ldots, x_n\}$ and some binary retracts Q, R we have:





neglecting the order of variables. The choice of variables $y_1, z_2, \ldots, z_k$ contradicts the irreducibility of E.

Impossibility of all the cases (1)-(6) leads to conclusion that cannot be card $A^{\sim} > \bar{n}$. Lemma is proved.

Lemma 3. *For every equivalence class $A^{\sim}$ there exists a quasigroup $Q_{A^{\sim}}$ such that every $B \in A^{\sim}$ is diisotopic with $Q_{A^{\sim}}$ and diisotopy is of the form:*

$$B(y_1, \ldots, y_k) = \langle \cdot, B \rangle^{-1} Q_{A^{\sim}}^{\pi_B}(\langle \cdot, B \rangle B_{\{1\}} y_1, \ldots, \langle \cdot, B \rangle B_{\{k\}} y_k).$$

Proof: From lemma 2 it follows that A is either binary operation or $A^{\sim}$ has exactly $\bar{n}$ elements.

(1) Let A be a k-ary operation $(k>2)$ and

$$Q_{A^{\sim}}(y_1, \ldots, y_k) = \langle \cdot, A \rangle A(A_{\{1\}}^{-1} \langle \cdot, A \rangle^{-1} y_1, \ldots, A_{\{k\}}^{-1} \langle \cdot, A \rangle^{-1} y_k).$$

Let $B \in A^{\sim}$. Then, if $B=A$ then:

$$A(y_1, \ldots, y_k) = \langle \cdot, A \rangle^{-1} Q_{A^{\sim}}^{\pi_A}(\langle \cdot, A \rangle A_{\{1\}} y_1, \ldots, \langle \cdot, A \rangle A_{\{k\}} y_k)$$

where π_A is identity permutation of $\{1, \ldots, k\}$. *If* $B \neq A$ then, equalizing terms containing A and B and for $M = \{i_r \mid x_{i_r} \in f_A(r),\ 1 \leqslant r \leqslant k\}$ we have:

$$\langle \cdot, B\rangle B^{\overset{-1}{\pi_B}}(\langle B, x_{i_1}\rangle x_{i_1}, \ldots, \langle B, x_{i_k}\rangle x_{i_k}) =$$

$$= \langle \cdot, A\rangle A(\langle A, x_{i_1}\rangle x_{i_1}, \ldots, \langle A, x_{i_k}\rangle x_{i_k}) =$$

$$= Q_{A^\sim}^{\pi_A}(\langle \cdot, A\rangle A_{\{1\}} \langle A, x_{i_1}\rangle x_{i_1}, \ldots, \langle \cdot, A\rangle A_{\{k\}} \langle A, x_{i_k}\rangle x_{i_k}) =$$

$$= Q_{A^\sim}(\langle \cdot, B\rangle B_{\{1\}} \langle B, x_{i_1}\rangle x_{i_1}, \ldots, \langle \cdot, B\rangle B_{\{k\}} \langle B, x_{i_k}\rangle x_{i_k}).$$

It follows that:

$$B(y_1, \ldots, y_k) = \langle \cdot, B\rangle^{-1} Q_{A^\sim}^{\pi_B}(\langle \cdot, B\rangle B_{\{1\}} y_1, \ldots, \langle \cdot, B\rangle B_{\{k\}} y_k).$$

(2) Let A be a binary operation and

$$Q_{A^\sim}(y_1, y_2) = \langle \cdot, A\rangle A(\overset{-1}{A_{\{1\}}} \langle \cdot, A\rangle^{-1} y_1, \overset{-1}{A_{\{2\}}} \langle \cdot, A\rangle^{-1} y_2)$$

The proof of the case (2) is by induction on length of one of shortest sequences $A = B_1, \ldots, B_m = B$ such that for all r $(1 \leqslant r < m)$ $B_r \leftrightarrow B_{r+1}$. For $m = 1$, $B = A$ and we have:

$$A(y_1, y_2) = \langle \cdot, A\rangle^{-1} Q_{A^\sim}(\langle \cdot, A\rangle A_{\{1\}} y_1, \langle \cdot, A\rangle A_{\{2\}} y_2)$$

Let us suppose that lemma holds for all sequences of length less than m. Then we have for $C = B_{m-1}$:

$$C(y_1, y_1) = \langle \cdot, C\rangle^{-1} Q_{A^\sim}^{\pi_C}(\langle \cdot, A\rangle C_{\{1\}} y_1, \langle \cdot, C\rangle C_{\{2\}} y_2).$$

From $B_m = B$ it follows that $B \leftrightarrow C$ and for some $x_i \in f_B(1)$ and $x_j \in f_B(2)$ we have:

$$\langle \cdot, B\rangle B(\langle B, x_i\rangle x_i, \langle B, x_j\rangle x_j) =$$

$$= \langle \cdot, C\rangle C^{\sigma_C}(\langle C, x_i\rangle x_i, \langle C, x_j\rangle x_j) =$$

$$= \langle \cdot, C\rangle \langle \cdot, C\rangle^{-1} Q_{A^\sim}^{\pi_C \sigma_C}(\langle \cdot, C\rangle C' \langle C, x_i\rangle x_i, \langle \cdot, C\rangle C'' \langle C, x_j\rangle_j) =$$

$$= Q_{A^\sim}^{\pi_C \sigma_C}(\langle \cdot, B\rangle B_{\{1\}} \langle B, x_i\rangle x_i, \langle \cdot, B\rangle B_{\{2\}} \langle B, x_j\rangle x_j),$$

where $\{C', C''\} = \{C_{\{1\}}, C_{\{2\}}\}$. It follows that

$$B(y_1, y_2) = \langle \cdot, B\rangle^{-1} Q_{A^\sim}^{\pi_B}(\langle \cdot, B\rangle B_{\{1\}} y_1, \langle \cdot, B\rangle B_{\{2\}} y_2)$$

where $\pi_B = \pi_C \sigma_C$.

Lemma 4. *Let* $A < B$ *and* $\urcorner A \sim B$. *If* $A \leftrightarrow C$, $B \leftrightarrow D$ *and* C, D *are comparable then* $C < D$.

Proof: From $A \leftrightarrow C$, $B \leftrightarrow D$ it follows that cannot be $C = D$. From comparability then follows that either $C < D$ or $D < C$ holds.

Suppose that $D<C$ holds. Then there exist variables y_1, y_2, y_3 such that $A(B(y_1, y_2), y_3) = D(C(y_1, y_2), y_3)$ or $A(B(y_1, y_2), y_3) = D(y_1, C(y_2, y_3))$ neglecting the order of variables. In the first case we have $B \leftrightarrow C$ which leads to conclusion that $A \sim B$, which is impossible. In the second case we have $B \leftrightarrow D$ and $A \leftrightarrow D$ contrary to hypothesis $\neg A \sim B$.

It cannot be but $C<D$ which had to be proved.

Lemma 5. *Let B, C be $\sim$-related, incomparable operations occuring in the term t and let $A = \inf(B, C)$ with respect to the ordering $\leqslant_t$. Then $A \sim B$.*

Proof: Let D, E, F be operations occuring in t, such that $A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$ while not simultaneously $D<E$, $D<F$ and E and F incomparable. Such operations we can always find, otherwise cannot be $B \sim C$.

Let $\neg A \sim B$. From $A<C$, $A \leftrightarrow D$, $C \leftrightarrow F$, according to lemma 4, it follows that $D<E$. From $\neg A \sim B$, $B \sim C$ we conclude $\neg A \sim C$. From $A<C$, $A \leftrightarrow D$, $C \leftrightarrow F$, according to lemma 4 it follows that $D<F$. Then E, F must be comparable.

Let $E<F$. Then there exists a set of variables $\{y_1, \ldots, y_4\} = \{z_1, \ldots, z_4\}$ such that:

$$A(B(y_1, y_2), C(y_3, y_4)) = D(E(F(z_1, z_2), z_3), z_4)$$

neglecting the order of arguments.

If $\{z_1, z_2\} = \{y_1, y_3\}$, $\{z_1, z_2\} = \{y_1, y_4\}$, $\{z_1, z_2\} = \{y_2, y_3\}$ or $\{z_1, z_2\} = \{y_2, y_4\}$ then $A \leftrightarrow F$ and from $C \leftrightarrow F$ and $B \sim C$ we must conclude that $A \sim B$.

If $\{z_1, z_2\} = \{y_1, y_2\}$ then $z_3 \in \{y_3, y_4\}$ and $A \leftrightarrow E$ and from $B \leftrightarrow E$ we conclude that $A \sim B$.

If $\{z_1, z_2\} = \{y_3, y_4\}$ then $z_3 \in \{y_1, y_2\}$ and $A \leftrightarrow E$ and $A \sim B$ as in the previous case.

In any case from $\neg A \sim B$ it follows that $A \sim B$, so $A \sim B$ always holds.

Lemma 6. *Let A, B, C be operations occuring in the same term such that $A \sim C$ and $A<B<C$. Then $A \sim B$.*

Lemma 6 is lemma 15 in [3]. Order of variables is not essential here. The only difference is that $\sim$ is not the symbol of isotopy as in [3], but rather of diisotopy.

Lemmas 5 and 6 show us that $St_i \cap A^{\sim}$ is always a convex set with respect to the ordering $\leqslant_{t_i}$.

Let $x_i \in V_A$ iff there exists $B \in A^{\sim}$ such that x_i is a subterm of TB. We can call V_A a set of A-variables.

Let $\equiv_A$ be a relation defined on V_A with: $x_i \equiv_A x_j$ iff for all $B \in A^{\sim}$ $\arg(B, x_i) = \arg(B, x_j)$. Relation $\equiv_A$ is a partial equivalence on the set of all variables of Γ.

Lemma 7. *Let A be an operation of a system Γ. Choose variables $y_1, \ldots, y_k$ in such a way that every equivalence class of $\equiv_A$ contains exactly* *one of variables $y_1, \ldots, y_k$. Substitute the other variables with corresponding elements of S. The system $\Gamma(A^{\sim})$ we obtain has a property that a nonunary operation B from Γ occurs in $\Gamma(A^{\sim})$ iff $B \sim A$.*

Proof is trivial if we keep in mind the definition of $\equiv_A$ and choice of $y_1, \ldots, y_k$.

System $\Gamma(A^{\sim})$ gives us a condition for $Q_{A^{\sim}}$.

Lemma 8. *An operation $Q_{A^{\sim}}$ is:*

(a) *a quasigroup (of corresponded arity)*

(b) *a group if in some term of Γ there are at least two operations from $A^{\sim}$* (card $A^{\sim} > \bar{n}$).

(c) *an abelian group if* card $A^{\sim} > \bar{n}$ *and no replacing of an operation by its dual, transforms $\Gamma(A^{\sim})$ in a system of the first kind.*

Proof: (a) Trivial.

(b) According to lemma 2 $Q^*_{A^{\sim}}$ is a binary operation and there are A-variables y_1, y_2, y_3 such that for some operations $B, C, D, E \in A^{\sim}$ the following equality holds:

$$P_1 B'(P_2 C'(P_3 y_1, P_4 y_2), P_5 y_3) = P_6 D'(P_7 y_1, P_8 E'(P_9 y_2, P_{10} y_3))$$

where $P_1, \ldots, P_{10}$ are compositions of some unary retracts from Γ and F' is F or F^* (dual of F). According to theorem on four quasigroups, $Q_{A^{\sim}}$ is a group.

(c) According to (b) $Q_{A^{\sim}}$ is a group and there are A-variables y_1, y_2 such that for some operations $B, C \in A^{\sim}$ the following equality holds:

$$\langle \cdot, B\rangle B(\langle B, y_1\rangle y_1, \langle B, y_2\rangle y_2) = \langle \cdot, C\rangle C(\langle C, y_2\rangle y_2, \langle C, y_1\rangle y_1)$$

where both B, C are isotopic to $Q_{A^{\sim}}$ or $Q^*_{A^{\sim}}$, which is always possible because $\Gamma(A^{\sim})$ is not equivalent to a system of the first kind. Applying lemma 3 we obtain:

$$\langle \cdot, B\rangle \langle \cdot, B\rangle^{-1} Q^{\pi_B}_{A^{\sim}}(\langle \cdot, B\rangle B_{\{1\}} \langle B, y_1\rangle y_1, \langle \cdot, B\rangle B_{\{2\}} \langle B, y_2\rangle y_2) =$$

$$= \langle \cdot, C\rangle \langle \cdot, C\rangle^{-1} Q^{\pi_C}_{A^{\sim}}(\langle \cdot, C\rangle C_{\{1\}} \langle C, y_2\rangle y_2, \langle \cdot, C\rangle C_{\{2\}} \langle C, y_1\rangle y_1)$$

where $\pi_B = \pi_C$, so we have:

$$Q_{A^{\sim}}(\langle \cdot, B\rangle B_{\{1\}} \langle B, y_1\rangle y_1, \langle \cdot, B\rangle B_{\{2\}} \langle B, y_2\rangle y_2) =$$

$$= Q_{A^{\sim}}(\langle \cdot, C\rangle C_{\{1\}} \langle C, y_2\rangle y_2, \langle \cdot, C\rangle C_{\{2\}} \langle C, y_1\rangle y_1).$$

$$\langle \cdot, B\rangle B_{\{1\}} \langle B, y_1\rangle y_1 = \langle \cdot, C\rangle C_{\{2\}} \langle C, y_1\rangle y_1$$

and

$$\langle \cdot, B\rangle B_{\{2\}} \langle B, y_2\rangle y_2 = \langle \cdot, C\rangle C_{\{1\}} \langle C, y_2\rangle y_2$$

are consequences of Γ so finally we obtain $Q_{A^{\sim}}(z_1\ z_2) = Q_{A^{\sim}}(z_2, z_1)$ i.e. $Q_{A^{\sim}}$ is an abelian group.

Theorem *Let* Γ *be an irreducible system of balanced general functional equations on quasigroups, of extended equality type. The general solution of* Γ *is given by:*

$$(*)\qquad A(x_1, \dots, x_k) = \langle \cdot, A\rangle^{-1}\, Q_{A^\sim}^{\pi_A}(\langle \cdot, A\rangle A_{\{1\}} x_1, \dots, \langle \cdot, A\rangle A_{\{k\}} x_k)$$

where (for any operation B *occuring in* Γ*):*

(1) S *is any nonempty set*

(2) $Q_{B^\sim} = Q_{A^\sim}$ *for* $B \sim A$ *(*$B \sim A$ *iff diisotopy of* B *and* A *is a consequence of* Γ*)*

(3) *all permutations* π_B $(B \sim A)$ *on* $\{1, \dots, k\}$ *are uniquely determined if one of them is given.*

(4) $Q_{A^\sim}$ *is any k-ary quasigroup on* S *iff in every term of* Γ *occurs exactly one operation from* $A^\sim$.

(5) $Q_{A^\sim}$ *is any binary group on* S *iff in some term of* Γ *occurs at least two operations from* $A^\sim$.

(6) $Q_{A^\sim}$ *is any binary abelian group on* S *iff no exchange of some opera-ions occuring in* $\Gamma(A^\sim)$ *with dual operations, transforms* $\Gamma(A^\sim)$ *in a system of he first kind.*

$$(7)\qquad \dots, A_{\{1\}}, \dots, A_{\{k\}}, \dots$$

are permutations on S *for which the following equations hold:*

$$(7_i)\qquad \langle \cdot, x_i\rangle_1 = \cdots = \langle \cdot, x_i\rangle_{\bar{n}} \qquad (i = 1, \dots, n)$$

where n *is the number of variables occuring in* Γ *and* $\langle \cdot, x_i\rangle_r$ *is a composition* $\langle \cdot, x_i\rangle$ *in the term* t_r.

Proof: (a) It can be easily checked that (*) with (1)—(7) is a solution of Γ.

(b) From lemmas 3 and 8 we deduce that (*) gives a general solution of Γ. Formulas (7_i) $(i = 1, \dots, n)$ are consequences of Γ.

Arbitrariness of S follows from the fact that on any nonempty set, we can define operations (as in the proof of consistency of Γ) which are particular solution of Γ.

Example 1. Let the following system Γ be given:

$$\begin{aligned} &A(x_1, B(C(x_2, D(x_3, E(F(x_4, x_5), x_6)), x_7), x_8)) = \\ &= G(H(I(J(K(x_4, x_5), L(x_3, x_6)), x_7, x_2), x_1), x_8) = \\ &= M(x_8, N(x_1, P(x_7, x_2, Q(x_4, R(S(x_5, x_3), x_6))))) \end{aligned}$$

Sets $\{A, B, G, H, M, N\}$, $\{C, I, P\}$, $\{D, E, F, J, K, L, Q, R, S\}$ are equivalence classes of $\sim$. The general solution of Γ is given by:

$$A(x, y) = A_{\{1\}} x * A_{\{2\}} y$$

$$B(x, y) = A_{\{2\}}^{-1}(A_{\{2\}} B_{\{1\}} x * A_{\{2\}} B_{\{2\}} y)$$

$C(x, y, z) = (A_{\{2\}} B_{\{1\}})^{-1} T(A_{\{2\}} B_{\{1\}} C_{\{1\}} x, A_{\{2\}} B_{\{1\}} C_{\{2\}} y, A_{\{2\}} B_{\{1\}} C_{\{3\}} z)$

$D(x, y) = (A_{\{2\}} B_{\{1\}} C_{\{2\}})^{-1} (A_{\{2\}} B_{\{1\}} C_{\{2\}} D_{\{1\}} x + A_{\{2\}} B_{\{1\}} C_{\{2\}} D_{\{2\}} y)$

$E(x, y) = (A_{\{2\}} B_{\{1\}} C_{\{2\}} D_{\{2\}})^{-1} (A_{\{2\}} B_{\{1\}} C_{\{2\}} D_{\{2\}} E_{\{1\}} x + A_{\{2\}} B_{\{1\}} C_{\{2\}} D_{\{2\}} E_{\{2\}} y)$

$F(x, y) = (A_{\{2\}} B_{\{1\}} C_{\{2\}} D_{\{2\}} E_{\{1\}})^{-1} (A_{\{2\}} B_{\{1\}} C_{\{2\}} D_{\{2\}} E_{\{1\}} F_{\{1\}} x +$

$+ A_{\{2\}} B_{\{1\}} C_{\{2\}} D_{\{2\}} E_{\{1\}} F_{\{2\}} y)$

$G(x, y) = G_{\{1\}} x * G_{\{2\}} y$

$H(x, y) = G_{\{1\}}^{-1} (G_{\{1\}} H_{\{2\}} y * G_{\{1\}} H_{\{1\}} x)$

$I(x, y, z) = (G_{\{1\}} H_{\{1\}})^{-1} T(G_{\{1\}} H_{\{1\}} I_{\{3\}} z, G_{\{1\}} H_{\{1\}} I_{\{1\}} x, G_{\{1\}} H_{\{1\}} I_{\{2\}} y)$

$J(x, y) = (G_{\{1\}} H_{\{1\}} I_{\{1\}})^{-1} (G_{\{1\}} H_{\{1\}} I_{\{1\}} J_{\{1\}} x + G_{\{1\}} H_{\{1\}} I_{\{1\}} J_{\{2\}} y)$

$K(x, y) = (G_{\{1\}} H_{\{1\}} I_{\{1\}} J_{\{1\}})^{-1} (G_{\{1\}} H_{\{1\}} I_{\{1\}} J_{\{1\}} K_{\{1\}} x + G_{\{1\}} H_{\{1\}} I_{\{1\}} J_{\{1\}} K_{\{2\}} y)$

$L(x, y) = (G_{\{1\}} H_{\{1\}} I_{\{1\}} J_{\{2\}})^{-1} (G_{\{1\}} H_{\{1\}} I_{\{1\}} J_{\{2\}} L_{\{1\}} x + G_{\{1\}} H_{\{1\}} I_{\{1\}} J_{\{2\}} L_{\{2\}} y)$

$M(x, y) = M_{\{2\}} y * M_{\{1\}} x$

$N(x, y) = M_{\{2\}}^{-1} (M_{\{2\}} N_{\{1\}} x * M_{\{2\}} N_{\{2\}} y)$

$P(x, y, z) = (M_{\{2\}} N_{\{2\}})^{-1} T(M_{\{2\}} N_{\{2\}} P_{\{2\}} y, M_{\{2\}} N_{\{2\}} P_{\{3\}} z, M_{\{2\}} N_{\{2\}} P_{\{1\}} x)$

$Q(x, y) = (M_{\{2\}} N_{\{2\}} P_{\{3\}})^{-1} (M_{\{2\}} N_{\{2\}} P_{\{3\}} Q_{\{1\}} x + M_{\{2\}} N_{\{2\}} P_{\{3\}} Q_{\{2\}} y)$

$R(x, y) = (M_{\{2\}} N_{\{2\}} P_{\{3\}} Q_{\{2\}})^{-1} (M_{\{2\}} N_{\{2\}} P_{\{3\}} Q_{\{2\}} R_{\{1\}} x + M_{\{2\}} N_{\{2\}} P_{\{3\}} Q_{\{2\}} R_{\{2\}} y$

$S(x, y) = (M_{\{2\}} N_{\{2\}} P_{\{3\}} Q_{\{2\}} R_{\{1\}})^{-1} (M_{\{2\}} N_{\{2\}} P_{\{3\}} Q_{\{2\}} R_{\{1\}} S_{\{1\}} x +$

$+ M_{\{2\}} N_{\{2\}} P_{\{3\}} Q_{\{2\}} R_{\{1\}} S_{\{2\}} y)$

where $A_{\{1\}}, \ldots, S_{\{2\}}$ are arbitrary permutations for which the following equations hold:

$A_{\{1\}} = G_{\{1\}} H_{\{2\}} = M_{\{2\}} N_{\{1\}}$

$A_{\{2\}} B_{\{1\}} C_{\{1\}} = G_{\{1\}} H_{\{1\}} I_{\{3\}} = M_{\{2\}} N_{\{2\}} P_{\{2\}}$

$A_{\{2\}} B_{\{1\}} C_{\{2\}} D_{\{1\}} = G_{\{1\}} H_{\{1\}} I_{\{1\}} J_{\{2\}} L_{\{1\}} = M_{\{2\}} N_{\{2\}} P_{\{3\}} Q_{\{2\}} R_{\{1\}} S_{\{2\}}$

$A_{\{2\}} B_{\{1\}} C_{\{2\}} D_{\{2\}} E_{\{1\}} F_{\{1\}} = G_{\{1\}} H_{\{1\}} I_{\{1\}} J_{\{1\}} K_{\{1\}} = M_{\{2\}} N_{\{2\}} P_{\{3\}} Q_{\{1\}}$

$A_{\{2\}} B_{\{1\}} C_{\{2\}} D_{\{2\}} E_{\{1\}} F_{\{2\}} = G_{\{1\}} H_{\{1\}} I_{\{1\}} J_{\{1\}} K_{\{2\}} = M_{\{2\}} N_{\{2\}} P_{\{3\}} Q_{\{2\}} R_{\{1\}} S_{\{1\}}$

$A_{\{2\}} B_{\{1\}} C_{\{2\}} D_{\{2\}} E_{\{2\}} = G_{\{1\}} H_{\{1\}} I_{\{1\}} J_{\{2\}} L_{\{2\}} = M_{\{2\}} N_{\{2\}} P_{\{3\}} Q_{\{2\}} R_{\{2\}}$

$A_{\{2\}} B_{\{1\}} C_{\{3\}} = G_{\{1\}} H_{\{1\}} I_{\{2\}} = M_{\{2\}} N_{\{2\}} P_{\{1\}}$

$A_{\{2\}} B_{\{2\}} = G_{\{2\}} = M_{\{1\}}$

$*$ is an arbitrary group, T an arbitrary 3-quasigroup and $+$ an arbitrary abelian group. All of them are defined on a given nonempty set S.

Example 2. If all operations of Γ are binary and system contains only one equation, we obtain results of BELOUSOV and LIVŠIC on solutions of functional equations of the second kind for binary quasigroups.

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