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ON THE ASYMPTOTIC FORMULAS FOR POWERFUL NUMBERS

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1. Introduction

A natural number is said to be powerful if it contains powers oi primes as factors; more precisely let $G(k)$ denote for each fixed natural $k>1$ the set of all natural numbers with the property that if a prime p divides an element of $G(k)$ then p^k divides it also. In other words the set of powerful numbers $G(k)$ contains for a fixed k numbers whose canonical factorisation is

$$(1) \qquad n=p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_i^{\alpha_i} \qquad \alpha_j \geq k \text{ for } j=1, 2, \ldots, i.$$

For each α_j there is a uniquely determined β_j such that $\alpha_j \equiv \beta_j$ (modn) and $k \leq \beta_j \leq 2k-1$, so that every powerful number may be written uniquely in the form

$$(1.2) \qquad n=a_o^k a_1^{k+1} a_2^{k+2} \cdots a_{k-1}^{2k-1}$$

where $a_1, a_2, \ldots, a_{k-1}$ are squarefree numbers and $(a_i, a_j)=1$ for $1 \leq i<j \leq k-1$. Thus if we set

$$f_k(n)=\begin{cases} 1 & n \in G(k) \\ 0 & n \notin G(k) \end{cases}, \quad F_k(s)=\sum_{n=1}^{\infty} f_k(n) n^{-s},$$

it follows that

$$(1.3) \qquad F_k(s)=\prod_p (1+p^{-ks}+p^{-(k+1)s}+\cdots)=\prod (1+p^{(1-k)s}(p^s-1)^{-1}), \text{ or}$$

$$(1.4) \qquad F_k(s)=\prod_p (1+p^{-(k+1)s}+p^{-(k+2)s}+\cdots+p^{-(2k-1)s}) \zeta(ks),$$

where for $\operatorname{Re} s>1$ $\zeta(s)=\prod_p (1-p^{-s})^{-1}$ is Riemann's zeta function. Further factoring gives

$$\frac{\zeta(ks)\zeta((k+1)s)\zeta((k+2)s)\cdots\zeta((2k-1)s)}{F_k(s)\zeta((2k+2)s)\zeta((2k+4)s)\cdots\zeta((4k-2)s)}=$$

$$=\prod_p \frac{(1+p^{-(k+1)s})\,(1+p^{-(k+2)s})\cdots(1+p^{-(2k-1)s})}{(1+p^{-(k+1)s}+p^{-(k+2)s}+\cdots+p^{-(2k-1)s})}=$$

$$=\prod_p\left(1+\frac{p^{-(2k+3)s}+p^{-(2k+4)s}+2p^{-(2k+5)s}+\cdots+p^{-(3k^2-3k)s/2}}{1+p^{-(k+1)s}+p^{-(k+2)s}+\cdots+p^{-(2k-1)s}}\right)=$$

$$=\prod_p (1+a_k(p,s)),$$

where $a_k(p,s)\sim p^{-(2k+3)s}$ for real s as $p\to\infty$, so that

$$F_k(s)=\frac{\zeta(ks)\zeta((k+1)s)\zeta((k+2))s\cdots\zeta((2k-1)s)}{\zeta((2k+2)s)\zeta((2k+4)s\cdots\zeta((4k-2)s)}\varphi_k(s), \tag{1.5}$$

where $\varphi_k(s)=\prod_p (1+a_k(p,s))^{-1}$ has a Dirichlet series with the abscissa of convergence $1/(2k+3)$ for $k>2$ ($\varphi_k(s)=1$ for $k=2$), so that we may write

$$F_k(s)=G_k(s)H_k(s), \tag{1.6}$$

where

$$H_k(s)=\sum_{n=1}^{\infty} h_k(n)n^{-s}=\zeta(ks)\zeta((k+1)s)\cdots\zeta((2k-1)s)$$

and

$$G_k(s)=\sum_{n=1}^{\infty} g_k(n)n^{-s}=\varphi_k(s)\zeta^{-1}((2k+2)s)\cdots\zeta^{-1}((4k-2)s)$$

is a Dirichlet series with the abscissa of convergence equal to $1/(2k+2)$. If $A_k(x)$ denotes the number of powerful numbers not exceeding x for a fixed $k>2$, then we have

$$A_k(x)=\sum_{n\le x,\, n\in G(k)} 1=\sum_{n\le x} f_k(n)=\sum_{mn\le x} g_k(m)h_k(n). \tag{1.7}$$

Powerful numbers were first investigated by P. Erdös and G. Szekeres in [2]. Later researches include [3], [4], [6] and P. Bateman's and E. Grosswald's paper [1] where the following results were obtained:

$$A_2(x)=(\zeta(3/2)/\zeta(3))\,x^{1/2}+(\zeta(2/3)/\zeta(2))\,x^{1/3}+O(x^{1/6}\exp(-a\,\omega(x))), \tag{1.8}$$

where $a>0$ and $\omega(x)=(\log x)^{4/7}\cdot\log\log x)^{-3/7}$,

$$A_3(x)=\gamma_{03}\,x^{1/3}+\gamma_{13}\,x^{1/4}+\gamma_{23}\,x^{1/5}+O(x^{7/46}) \tag{1.9}$$

where $\gamma_{03}, \gamma_{13}, \gamma_{23}$ are computable constants,

$$(1.10) \qquad A_k(x) = \gamma_{ok} x^{1/k} + \gamma_{1k} x^{1/(k+1)} + \cdots + \gamma_{rk} x^{1/(k+r)} + \Delta_k(x),$$

where r is an integer >1, $r^2/2 < k \leq (r+1)^2/2$, γ_{ik} for $i=0, 1, \ldots, r$ are computable constants and

$$\Delta_k(x) = \begin{cases} O(x^{r/(kr+2k)} \log^r x) & r^2/2 < k < r(r+1)/2 \\ O(x^{1/(k+r+1)} \log^{r+1} x) & k = r(r+1)/2. \\ O(x^{1/(k+r+1)}) & r(r+1)/2 < k \leq (r+1)^2/2 \end{cases}$$

The main purpose of this paper is to give estimates for $A_k(x)$ and $\sum_{n \leq x} \alpha_k(n)$, where $\alpha_k(n)$ denotes the number of divisors of n from $G(k)$. I conjecture that for all $k > 2$ the following formula holds:

$$(1.11) \qquad A_k(x) = \gamma_{ok} x^{1/k} + \gamma_{1k} x^{1/(k+1)} + \cdots + \gamma_{k-1,k} x^{1/(2k-1)} + O(x^{\theta_k}),$$

where $\gamma_{ik} = \operatorname*{Res}_{s=1/(k+i)} F_k(s) s^{-1}$ for $i=0, 1, \ldots, k-1$ and $\theta_k < 1/(2k-1)$. This is proved with $\theta_k = 1/(2k) + \varepsilon$ for every $\varepsilon > 0$ under the truth of the Lindelöf hypothesis, and unconditionally for $k < 6$ with explicit values of θ_3, θ_4 and θ_5.

2. Estimates of $A_k(x)$.

If one uses Bateman's and Grosswald's proof of (1.8) and the estimate $M(x) = \sum_{n \leq x} \mu(n) = O(x \cdot \exp(-c\eta(x)))$, where $c > 0$ and $\eta(x) = (\log x)^{3/5} \cdot (\log \log x)^{-1/5}$, which is due to A. Walfisz [12], then in (1.8) the error term may be replaced by $O(x^{1/6} \cdot \exp(-b\eta(x)))$ where $b > 0$.

It is seen from (1.7) that $A_k(x)$ depends on $\sum_{n \leq x} h_k(n) = \sum_{n_o^k n_1^{k+1} \cdots n_{k-1}^{2k-1} \leq x} 1$, so that one may hope that a sufficiently sharp estimate of $\sum_{n \leq x} h_k(n)$ may lead to the conjectured formula (1.11) for $A_k(x)$.This is indeed so, as shown by the following

Lemma 1. *Suppose that for* $1/(2k+2) < \eta_k < 1/(2k-1)$ *we have*

$$(2.1) \qquad \sum_{n \leq x} h_k(n) = \sum_{n_o^k n_1^{k+1} \cdots n_{k-1}^{2k-1} \leq x} 1 = \sum_{i=k}^{2k-1} \Big(\prod_{j=k,\, j \neq i}^{2k-1} \zeta(j/i) \Big) x^{1/i} + O(x^{\eta_k}).$$

Then (1.11) *holds with* $\theta_k = \eta_k$.

Proof. $A_k(x) = \sum_{n \leq x} f_k(n) = \sum_{n \leq x} g_k(n) \sum_{m \leq x/n} h_k(m) =$

$$= \sum_{n \leq x} g_k(n) \sum_{i=k}^{2k-1} c_i x^{1/i} n^{-1/i} + O\Big(x^{\eta_k} \sum_{n \leq x} |g_k(n)| n^{-\eta_k}\Big),$$

where $c_i = \prod_{j=k, j\neq i}^{2k-1} \zeta(j/i)$. Since the abscissa of convergence of $G_k(s)$ equals $1/(2k+2)$ we have $\sum_{n\leq x} |g_k(n)| n^{-\eta_k} = O(1)$, and partial summation gives for every $\varepsilon > 0$ and $i = k, k+1, \ldots, 2k-1$

$$\sum_{n\leq x} g_k(n) n^{-1/i} = G_k(1/i) + O(x^{1/(2k+2)+\varepsilon-1/i}),$$

so that finally

$$A_k(x) = \sum_{i=k}^{2k-1} c_i G_k(1/i) x^{1/i} + \sum_{i=k}^{2k-1} O(x^{1/(2k+2)+\varepsilon}) + O(x^{\eta_k}),$$

which proves (1.11) with $\theta_k = \eta_k$ and $\gamma_{i-k, k} = c_i G(1/i)$ for $i = k, k+1, \ldots, 2k-1$, since ε may be taken arbitrarily small.

Theorem 1. *If the Lindelöf hypothesis is true, then for* $k \geq 3$

$$A_k(x) = \gamma_{0k} x^{1/k} + \gamma_{1k} x^{1/(k+1)} + \cdots + \gamma_{k-1, k} x^{1/(2k-1)} + O(x^{\theta_k}),$$

where for every $\varepsilon > 0$ $\theta_k = 1/(2k) + \varepsilon$, *that is,* (1.11) *holds.*

Proof. The Lindelöf hypothesis is (see [11], Ch. XIII) that for every $\varepsilon > 0$ $\zeta(1/2 + it) = O(t^{\varepsilon})$, and the best estimate known so far is G. A. Kolesnik's $\zeta(1/2 + it) = O(t^{173/1067} \log^{331/200} t)$, proved in 1973. It is known that the Riemann hypothesis that all non-trivial zeros of the zeta function have the real part 1/2 implies the Lindelöf hypothesis, so that (1.11) holds then under the Riemann hypothesis also.

By lemma 1 it is sufficient to prove (2.1) with $\eta_k = 1/(2k) + \varepsilon$. The classical method of contour integration is applied to the function $H_k(s)$ which is regular except for the esimple poles at $s = 1/k, 1/(k+1), \ldots, 1/(2k-1)$. Therefore if x is half a large odd integer and $b > 1/k$, then

$$\sum_{n\leq x} h_k n) = \int_{b-iT}^{b+iT} (2\pi i)^{-1} H_k(s) x^s s^{-1} \, ds + O(x^b T^{-1} (b - 1/k)^{-1}) + + O(\Phi(2x) x T^{-1} \log x). \tag{2.2}$$

For a proof of (2.2), see for example [7]. Here $\Phi(x)$ stands for a non-decreasing positive function for which $h_k(n) = O(\Phi(n))$. Since $H_k(s) = \zeta(ks) \zeta((k+ +1)s) \cdots \zeta((2k-1)s)$ we may take $\Phi(x) = x^{\varepsilon}$ for every $\varepsilon > 0$. Therefore for fixed $1 > b > 1/k$ and $\varepsilon > 0$

$$\sum_{n\leq x} h_k(n) = (2\pi i)^{-1} \int_{b-iT}^{b+iT} H_k(s) x^s s^{-1} \, ds + O(x^{1+\varepsilon} T^{-1}). \tag{2.3}$$

The classical residue theorem gives then

$$(2\pi i)^{-1} \int_{b-iT}^{b+iT} H_k(s) x^s s^{-1} \, ds = \sum_{i=k}^{2k-1} \operatorname*{Res}_{s=1/i} H_k(s) x^s s^{-1} + (2\pi i)^{-1} (I_1 + I_2 + I_3),$$

where for $a<1/(2k-1)$ we have

$$I_1=\int_{a-iT}^{a+iT} H_k(s)\,x^s s^{-1}\,ds,\quad I_2=\int_{b-iT}^{a-iT} H_k(s)\,x^s s^{-1}\,ds,\quad I_3=\int_{a+iT}^{b+iT} H_k(s)\,x^s s^{-1}\,ds.$$

If we choose $a=1/2k$ then the Lindelöf hypothesis gives $H_k(s)=O(t^{k\varepsilon})$ for every $\varepsilon>0$ when s lies on the segments $(a-iT,\ a+iT)$, $(b-iT,\ a-iT)$, $(a+iT,\ b+iT)$.

$$I_1=\int_{a-iT}^{a+iT} H_k(s)\,x^s s^{-1}\,ds=O(x^{1/2k})+O\Big(\int_1^T |H_k(1/(2k)+\mathrm{it})|\,x^{1/2k}t^{-1}\,dt\Big)=$$
$$=O(x^{1/2k}T^{k\varepsilon}).$$

$$I_2=O\Big(\int_a^b |H_k(t-iT)|\,x^t\,T^{-1}\,dt\Big)=O\Big(x^b T^{-1}\int_a^b t^{k\varepsilon}\,dt\Big)=O(x^b T^{-1}),$$

and the same estimate holds for I_3 since b is fixed. On the other hand if we set

$$d_i=\prod_{j=k,\ j\neq i}^{2k-1}\zeta(j/i),\ \text{then}$$

$$\sum_{i=k}^{2k-1}\operatorname*{Res}_{s=1/i} H_k(s)\,x^s s^{-1}=\sum_{i=k}^{2k-1} d_i\,x^{1/i},$$

$$\sum_{n\le x} h_k(n)=\sum_{i=k}^{2k-1} d_i\,x^{1/i}+O(x^{1+\varepsilon}T^{-1})+O(x^{1/2k}T^{k\varepsilon}). \tag{2.4}$$

If we choose $T=x^{(2k-1)/2k}$ we obtain the conclusion of the theorem, since the restriction that x is half an odd integer is clearly irrelevant.

On the other hand, it seems equally interesting to try to find all k for wh'ch (1.11) holds unconditionally, without unproved conjectures like Riemann's or Lindelöf's. In this direction the following theorem is proved, which gives somewhat better values than [6].

Theorem 2. *The asymptotic formula* (1.11) *holds for* $2<k<6$ *with* $\theta_3=655/4643$, $\theta_4=257/2072$ *and* $\theta_5=665613/6227997$.

Proof. The case $k=2$ is not being mentioned because of (1.8). For $k\geq 3$ we need the following result of H.-E. Richert [8], which says in a slightly different notation that if $\Psi(x)=x-[x]-1/2$, $\gamma>0$, $x,z>1$ and if $(\varkappa,\lambda)$ is any pair of exponents, then

$$\sum_{n\le z}\Psi(xn^{-\gamma})=O(x^{-1/2}z^{1+\gamma/2})+\begin{cases} O(x^{\varkappa/(\varkappa+1)}z^{(\lambda-\gamma\varkappa)/(\varkappa+1)}) & \lambda>\gamma\varkappa\\ O(x^{\varkappa/(\varkappa+1)}\log z) & \lambda=\gamma\varkappa\,.\\ O(x^{\varkappa/(1+\varkappa-\lambda+\gamma\varkappa)}) & \lambda<\gamma\varkappa\end{cases} \tag{2.5}$$

The definition of a pair of exponents (the theory was founded by Van der Corput, later simplified by Rankin and Phillips) is also given in [8]; let it be mentioned here only that if (k,l) is a pair of exponents then

$$A(k,l)=(k/(2k+2),\ 1/2+l/(2k+2))\ \text{and}\ B(k,l)=(l-1/2,\ k+1/2)$$

are also new pairs of exponents, (0, 1) is a pair of exponents, and for every pair of exponents (k, l) we have $0 \leq k \leq 1/2 \leq l \leq 1$.

The proof of theorem 2 follows closely the proof of Satz 7, p. 319 of E. Krätzel's paper [6], but a more general result than his Hilfsatz 1, p. 317 is proved here, which combined with his method of proof and another choice of the pair of exponents $(\varkappa, \lambda)$ leads to somewhat better values of θ_3, θ_4 and θ_5 than those given by Krätzel in [6], p. 324 ($\theta_k = \beta_k$ in his notation). Following [6] let

(2.6) $$\bar{a}_{k,m} = (k, k+1, \ldots, k+m), \quad d(\bar{a}_{k,m}; n) = \sum_{n_0^k n_1^{k+1} \cdots n_m^{k+m} = n} 1,$$

so that $d(\bar{a}_{k,m}; n)$ is the number of representation of n as $n = n_0^k n_1^{k+1} \cdots n_m^{k+m}$,

(2.7) $$D(\bar{a}_{k,m}; x) = \sum_{n \leq x} d(\bar{a}_{k,m}; n) = \sum_{i=k}^{k+m} \prod_{j=k, j \neq i}^{k+m} \zeta(j/i)\, x^{1/i} + \Delta(\bar{a}_{k,m}; x),$$

(2,8) $$\Delta(\bar{a}_{k,m}: x) = O(x^{\alpha_{k,m}}).$$

If $(k+m)\,\alpha_{k,m} < 1$, $1 \leq y \leq x$ then Krätzel proved in [6], p. 314

(2.9) $$\Delta\, \bar{a}_{k,m}; x) = -\sum_{n \leq x/y} d(\bar{a}_{k,m-1}; n)\, \Psi((x/n)^{1/(k+m)}) + O(y^{-1/(k+m)} (x/y)^{1/k}) + $$
$$+ O(y^{1/(k+m)} (x/y)^{\alpha_k{}^{m-1}}).$$

If for $m \geq 1$ we let $S_{k,m}(x, z) = \sum_{n \leq z} d(\bar{a}_{k,m-1}; n)\, \Psi((x/n)^{1/(k+m)})$ then instead of Hilfsatz 1 of [6], p. 317, the following more general result may be proved:

(2.10) $$S_{k,m}(x, z) = O(z^{1/k} (z/x)^{1/(2k+2m)}) + O(z^{\beta_{k,m}} (x/z)^{\varkappa/(\varkappa+1)(k+m)}),$$

where $(\varkappa, \lambda)$ is any pair of exponents, $\beta_{k,1} = \lambda/(\varkappa+1)\,k$ and for $m \geq 2$

$$\beta_{k,m} = (1 + \varkappa - \lambda(k+1)\,\beta_{k+1,m-1})/((2\varkappa + 2 - \lambda)\,k +$$
$$+ 1 + \varkappa - \lambda - (\varkappa+1)\,k\,(k+1)\,\beta_{k+1,m-1}).$$

To see that (2.10) holds we use induction on m and write

$$S_{k,m}(x, z) = S_1 + S_2 - S_3,$$

where $1 \leq y \leq 2$ and

$$S_1 = \sum_{n \leq y} d(\bar{a}_{k+1,m-2}; n) \sum_{m^k \leq z/n} \Psi((x/m^k n)^{1/(k+m)}),$$

$$S_2 = \sum_{m^k \leq z/y} \sum_{n \leq zm^{-k}} d(\bar{a}_{k+1,m-2}; n)\, \Psi((x/m^k n)^{1/(k+m)}) =$$
$$= \sum_{m^k \leq z/y} S_{k+1,m-2}(x/m^k, z/m^k),$$

$$S_3 = \sum_{n \leq y} d(\bar{a}_{k+1,m-2}; n) \sum_{m^k \leq z/y} \Psi((x/m^k)^{1/(k+m)}).$$

When $m=1$ (2.10) follows directly from (2.5), and when $m>1$ we use the induction hypothesis to obtain

$$S_{k,m}(x, z)=O\left(z^{1/k}(z/x)^{1/(2k+2m)}\right)+O\left(y^{1/(k+1)}(z/y)^{\lambda/(\varkappa+1)k}(x/z)^{\varkappa/(\varkappa+1)(k+m)}\right)+$$
$$+O\left((z/y)^{1/k}y^{\beta_{k+1,\,m-1}}(x/z)^{\varkappa/(\varkappa+1)(k+m)}\right).$$

The estimate (2.10) follows if y is chosen in such a way that the second and third O-term are equal. Putting (2.10) in (2.9) and choosing again y in such a may that two O-term are equal the following generalization of Satz 7 ([6], p. 319) is obtained:

If $(k+m)\,\alpha_{k,m-1}\leq 1$, then

(2.11) $\alpha_{k,m}=$

$$=\begin{cases}\dfrac{2+k\,\alpha_{k,m-1}}{5k+2m-2\,(k+m)\,\alpha_{k,m-1}} & \text{if } 3k\,(\varkappa+1)\,\beta_{k,m}\leq 2+(3\varkappa+1)\,k\,\alpha_{k,m-1}\\[2ex] \dfrac{(\varkappa+1)\,\beta_{k,m}-\varkappa\,\alpha_{k,m-1}}{1+(\varkappa+1)\,(k+m)\,(\beta_{k,m}-\alpha_{k,m-1})} & \text{if } 3k\,(\varkappa+1)\,\beta_{k,m}\geq 2+(3\varkappa+1)\,k\,\alpha_{k,m-1}\end{cases}.$$

From Euler's summation formula it follows for $z>0$ and $z\neq 1$

$$\sum_{n\leq x} n^{-z}=\zeta(z)+x^{1-z}(1-z)^{-1}-x^{-z}\,\Psi(x)+O(x^{-z-1}), \tag{2.12}$$

which for $1\leq a<b$ leads to the estimate (see [5], pp. 276—277 for details)

$$\sum_{m^a n^b\leq x} 1=\zeta(b/a)\,x^{1/a}+\zeta(a/b)\,x^{1/b}- \tag{2.13}$$
$$-\sum_{n^{a+b}\leq x}\left(\Psi((xn^{-b})^{1/a})+\Psi((xn^{-a})^{1/b})\right)+O(1).$$

Using (2.5) we obtain

$$\alpha_{k,1}=(\varkappa+\lambda)/(\varkappa+1)\,(2k+1), \tag{2.14}$$

and since from (2.10) we have $\beta_{k,2}=(1+\varkappa+\lambda)/(\varkappa+1)\,(2k+1)$ from (2.11) with the pair of exponents $(\varkappa, \lambda)=(11/30,\ 16/30)$ it follows

$$\alpha_{k,2}=\begin{cases}(191\,k+82)/(356\,k^2+425k+164) & \text{if } k\leq 6\\ 68/123\,(k+1) & \text{if } k>6\end{cases}. \tag{2.15}$$

From this estimate, which is sharper than Krätzel's (6.11) and (6.12) ([6], p. 320), we deduce with the help of (2.11)

$$\alpha_{3,2}=655/5643,\quad \alpha_{4,3}=257/2072,\quad \alpha_{5,4}=665613/6227997. \tag{2.16}$$

This proves the theorem, since by lemma 1 we may take $\theta_3=\alpha_{3,2}$, $\theta_4=\alpha_{4,3}$ and $\theta_5=\alpha_{5,4}$. Further improvements of the estimate for $\Delta(\bar{a}_{k,2};x)$ would give corresponding improvements of (2.16).

3. Estimates of $\sum_{n\le x} \alpha_k(n)$.

Let us now consider the divisor problem for elements of $G(k)$. If we define $\alpha_k(n) = \sum_{d\mid n,\, d\in G(k)} 1$, then $\alpha_k(n)$ represents the number of powerful divisors of n and the following theorem holds:

Theorem 3. *For* $k\ge 2$

$$\sum_{n\le x} \alpha_k(n) = F_k(1)\,x + C_0(k)\,x^{1/k} + C_1(k)^{1/(k+1)} + \Delta_k(x), \tag{3.1}$$

where for $i=0, 1$ *we have* $C_i(k) = \zeta(1/(k+i)) \lim_{s\to 1/(k+i)} F_k(s)((k+i)s-1)$, $z_3 = 8/35$, $z_4 = 28/149$, $z_k = (2k+7)/(2k^2+9k+13)$ *for* $k\ge 5$ *and*

$$\Delta_k(x) = \begin{cases} O(x^{105/407}\log^2 x) & k=2 \\ O(x^{z_k}) & k\ge 3 \end{cases}. \tag{3.2}$$

Proof. The error term $\Delta_k(x)$ will be seen to be of the same order of magnitude as the error term $\Delta(1, k, k+1; x)$ for $\sum_{n_0 n_1^k n_2^{k+1}\le x} 1$. The estimate of $\Delta(1, 2, 3; x)$ is closely related to the problem of enumerating finite non-isomorphic abelian groups (see [2] and [8]), and the estimate used here is due to B. R. Srinivasan [9]. An estimate for $\sum_{n\le x} \alpha_2(n)$ was obtained in [10] with a poorer error term which came from the estimate for $\Delta(1, 2, 3{:}\ x)$ given by Satz 6 of E. Krätzel's paper [5]; the values for z_3 and z_4 given here come from the estimates of $\Delta(1, 3, 4; x)$ and $\Delta(1, 4, 5; x)$ given by Satz 7, p. 826 of the same paper. For $k\ge 5$ we may derive a sharper estimate for $\Delta(1, k, k+1; x)$ than the one that follows from Satz 7 of [5]. To see this, observe that from (2.13) it follows

$$D(1, k; x) = \sum_{mn^k\le x} 1 = \zeta(k)\,x + \zeta(1/k)\,x^{1/k} - \sum_{n^{k+1}\le x} (\Psi((x/n)^{1/k}) + \Psi(xn^{-k})) + O(1)$$

and if (2.5) is used with $(\varkappa, \lambda) = (1/6,\ 4/6)$ we obtain $\Delta(1, k; x) = O(x^{1/(k+3)})$ for $k\ge 5$. Therefore if $k\ge 5$, $1\le y\le x$, $1\le z\le x$, then

$$\sum_{m^{k+1}\le y} \Delta(1, k; xm^{-k-1}) = O\Big(\sum_{m^{k+1}\le y} (xm^{-k-1})^{1/(k+3)}\Big) = O(x^{1/(k+3)}\,y^{2/(k+1)(k+3)}),$$

$$\sum_{n\le z} d(1, k; n)\,\Psi((x/n)^{1/(k+1)}) = O(z\cdot(z/x)^{1/(2k+2)}) + O(z^{4/9}\cdot(x/z)^{2/(9k+9)})$$

similarly to the way (2.10) was obtained, and since by Satz 1 of [5], p. 279 we have with $d(a, b; n) = \sum_{n_1^a n_2^b = n} 1$

$$\Delta(1, k, k+1; x) = \sum_{m^{k+1}\le y} \Delta(1, k; xm^{-k-1}) - \tag{3.3}$$

$$- \sum_{n\le x/y} d(1, k; n)\,\Psi((x/n)^{1/(k+1)}) + O(y^{1/(k+1)}) + O((x/y)\cdot y^{-1/(k+1)}),$$

we obtain finally

$$(3.4)\qquad \Delta(1, k, k+1; x) = O\left({}^{1/(k+3)} y^{2/(k+1)(k+3)}\right) + O\left(xy^{-(2k+3)/(2k+2)}\right) + + O\left((x/y)^{4/9} y^{2/(9k+9)}\right).$$

Let now in (3.4) $y = x^{2(k+1)(k+2)/(2k^2+9k+13)}$. Then the first two O-terms are equal, the third is of a smaller order of magnitude so that for $k \geq 5$ we have

$$(3.5)\qquad \Delta(1, k, k+1; x) = O\left(x^{(2k+7)/(2k^2+9k+13)}\right).$$

To prove the theorem note that $\alpha_k(n) = \sum_{d|n} f_k(d)$, so that we have

$$(3.6)\qquad V_k(s) = \sum_{n=1}^{\infty} \alpha_k(n) n^{-s} = \zeta(s) F_k(s) = B_k(s) C_k(s),$$

where (see (1.5)) for $k \geq 2$ $C_k(s) = \sum_{n=1}^{\infty} c_k(n) n^{-s} = \zeta(s)\zeta(ks)\zeta((k+1)s)$ and

$$B_k(s) = \sum_{n=1}^{\infty} b_k(n) n^{-s} =$$

$$= \begin{cases} 1/\zeta(6s) & k=2 \\ (\zeta(k+2)s) \cdots \zeta((2k-1)s)\varphi_k(s)/(\zeta((2k+2)s) \cdots \zeta((4k-2)s)) & k \geq 3 \end{cases}.$$

Therefore $\sum_{n \leq x} c_k(n) = D(1, k, k+1; x) = \sum_{n_0 n_1^k n_2^{k+1} \leq x} 1$, and for every $\varepsilon > 0$ $\sum_{n \leq x} |b_k(n)| = O(x^{1/(k+2)+\varepsilon})$, so that $\Delta(1, k, k+1; x) > O(x^{1/(k+2)+\varepsilon})$, and setting $\Delta_k(x) = \Delta(1, k, k+1; x)$ we obtain from (3.6) by the properties of Dirichlet series

$$\sum_{n \leq x} \alpha_k(n) = \sum_{n \leq x} \sum_{d|n} b_k(d) c_k(n/d) = \sum_{n \leq x} b_k(n) \sum_{m \leq x/n} c_k(m) =$$

$$\sum_{n \leq x} b_k(n) \left(\zeta(k)\zeta(k+1)(x/n) + \zeta(1/k)\zeta((k+1)/k)(x/n)^{1/k} + \right.$$

$$\left. + \zeta((1/(k+1))\zeta(k/(k+1))(x/n)^{1/(k+1)} + \Delta_k(x/n)\right) = \zeta(k)\zeta(k+1)\left(B_k(1) - \right.$$

$$\left. - \sum_{n > x} b_k(n) n^{-1}\right) \cdot x + \zeta(1/k)\zeta((k+1)/k)\left(B_k(1/k) - \sum_{n > x} b_k(n) n^{-1/k}\right) \cdot x^{1/k} +$$

$$+ \zeta(1/(k+1))\zeta(k/(k+1))\left(B_k(1/(k+1)) - \sum_{n > x} b_k(n) n^{-1/(k+1)}\right) \cdot x^{1/(k+1)} + O(\Delta_k(x)).$$

Partial summation gives for every $\varepsilon > 0$ and $y > 1/(k+2)$

$$(3.7)\qquad \sum_{n > x} b_k(n) n^{-y} = O\left(x^{1/(k+2)+\varepsilon-y}\right),$$

so that putting (3.7) into the preceding equation and collecting terms we obtain the conclusion of the theorem, since ε may be taken arbitrarily small.

REFERENCES

[1] P. T. Bateman, E. Grosswald, *On a theorem of Erdös and Szekeres*, Illinois Journal of Mathematics, 2 (1958), pp. 88—98.

[2] P. Erdös, G. Szekeres, *Über die Anzahl der Abelschen Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem*, Acta Sci. Math. Szeged, 7 (1935), pp. 95—102.

[3] S. W. Golomb, *Powerful numbers*, The Amer. Math. Monthly, 8 (1970), pp. 848—852.

[4] A. Ivić, *An asymptotic formula for elements of a semigroup of integers*, Matematički Vesnik, Belgrade 10 (25) Sv. 3, 1973, pp. 255—257.

[5] E. Krätzel, *Teiler probleme in drei Dimensionen*, Math. Nachrichten, 42 (1969), pp. 275—288.

[6] E. Krätzel, *Zahlen k-ter Art*, Amer. Journal of Math., 94 (1972) pp. 309—328.

[7] K. Prachar, *Primzahlvertreilung*, Springer Verlag, 1957.

[8] H. -E. Richert, *Über die Anzahl Abelscher Gruppen gegebener Ordnung I*, Math. Zeitschrift 56 (1), 1952, pp. 21—32.

[9] B. R. Srinivasan, *On the number of abelian groups of a given order*, Acta Arithmetica, 23 (1973), pp. 195—205.

[10] D. Suryanarayana, R. Sita Rama Chandra Rao, *The number of squarefull divisors of an integer*, Proc. Amer. Math. Soc., 34 (1), 1972, pp. 79—80.

[11] E. C. Titchmarsh, *The theory of the Riemann zeta function*, Oxford, 1951.

[12] A. Walfisz, *Weylsche Exponontialsummen in der neuren Zahlentheorie*, VEB Deutscher Verlag der Wiss., Berlin, 1963.

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