PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 23 (37) 1978, pp. 81—83

A THEOREM ON CONTRACTION MAPPINGS

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(Received June 10, 1977)

Meir and Keeler [4] defined weakly uniformly strict contraction mappings and proved a generalization of a fixed point theorem of Boyd and Wong [1]. In this note, we shall prove a fixed point theorem for mappings similar in [4] and give an extension of a recent result of Daneš [3].

Let (X, d) be a metric space and T a mapping of X into itself. For an $x \in X$, let $0(x)$ denote the orbit of x, that is $0(x)=\{T^n x : n \in I$ (nonnegative integers), $T^0 x = x\}$ and for a subset $A \subseteq X$, let $\delta(A)$ be the diameter of A and cl (A) the closure of A. Also, for $x, y \in X$, let

(1) $$M(x, y)=\max\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ d(x, Ty),\ d(y, Tx)\}.$$

Definition 1. The mapping $T: X \to X$ is a max-contraction on a subset S of X iff for each $\varepsilon > 0$ there exists a $\varepsilon_0 > \varepsilon$ and a $\delta_0 > 0$ such that

(2) $$\text{if } x, y \in S \text{ and } M(x, y) \leq \varepsilon + \delta_0 \text{ then } d(Tx, Ty) \leq \varepsilon_0.$$

Theorem. *Let* $T: X \to X$. *If for some* $x \in X$,

(i) $\delta(0(x)) < \infty$,

(ii) $\mathrm{cl}(0(x))$ *is complete in* X,

(iii) T *is a max-contraction on* $\mathrm{cl}(0(x))$.

Then T *has a unique fixed point in* $\mathrm{cl}(0(x))$.

Proof. Let for each $n \in I$, $x_n = T^n x$ and $\delta_n = \delta(0(x_n))$. Since for each $m, n \in I$, $n \leq m$, $0(x_m) \subseteq 0(x_n)$, it follows by (i that $\{\delta_n\}$ is a nonincreasing sequence of nonnegative reals and hence for some $\varepsilon \geq 0$,

(3) $$\delta_n \to \varepsilon.$$

We shall show that $\varepsilon = 0$. Suppose $\varepsilon > 0$. Then by (iii) there exists a ε_0, and a $\delta_0 > 0$ satisfying (2) on $\mathrm{cl}(0(x))$. Choose $N \in I$ such that $\delta_n \leq \varepsilon + \delta_0$ for all $n \geq N$. Now, if $m, n \in I$ and $m, n \geq N+1$, then since

$$\{x_{m-1}, x_{n-1}, Tx_{m-1}, Tx_{n-1}\} \subseteq 0(x\ \ z),$$

it follows by (1) that

$$M(x_{m-1}, X_{N-1}) \leq \delta_N \leq \varepsilon + \delta_0.$$

Consequently by (2)

$$d(x_m, x_n) \leq \varepsilon_0 < \varepsilon$$

for all $m, n \geq N+1$ and hence $\delta_{N+1} \leq \varepsilon_0 < \varepsilon$ contradicting the definition of ε. Thus $\varepsilon = 0$ and $\delta_n \to 0$. This implies that $\{T^n x\}$ is a Cauchy sequence. Therefore, by (ii) there exists a $u \in \mathrm{cl}(0(x))$ such that

$$T^n x \to u. \tag{4}$$

We shall now show that $u = Tu$. Suppose

$$d(u, Tu) = \varepsilon > 0. \tag{5}$$

Choose ε_0 and $\delta_0 > 0$ satisfying (2) on $\mathrm{cl}(0(x))$. Also, choose a $N \in I$ such that $\delta_n \leq \delta_0$ for all $n \geq N$. Since for any $n \in N$,

$$d(x_n, Tu) \leq d(x_n, u) + d(u, Tu), \text{ and}$$

$$d(Tx_n, u) \leq d(Tx_n, x_n) + d(x_n, u)$$

and $\delta_n = \delta(\mathrm{cl}(0(x_n)))$, it follows that for any $n \geq N$,

$$M(x_n, u) = \max\{d(x_n, u),\ d(x_n Tx_n),\ d(u, Tu)\, d(x_n, Tu),\ d(Tx_n, u)\} \leq \varepsilon + \delta_0$$

and consequently by (2)

$$d(Tx_n, Tu) \leq \varepsilon_0 < \varepsilon$$

for all $n \geq N$. Thus, it follows by (4) that $d(u, Tu) \leq \varepsilon_0 < \varepsilon$. This contradicts (5). Thus $Tu = u$. Now, suppose there are $u, v \in \mathrm{cl}(0(x))$ such that $Tu = u$ and $Tv = v$. Then $M(u, v) = M(Tu, Tv) = d(u, v)$ and hence by (2) $u = v$. Thus, u is the unique fixed point of T in $\mathrm{cl}(0(x))$.

A mapping $\Phi: R^+ = [0, \infty) \to [0, \infty)$ is right continuous at $t \geq 0$ if $t_n \geq t$, $t_n \to t$ then $\Phi(t_n) \to \Phi(t)$.

Definition (Daneš [3]). Let $\Phi: R^+ \to R^+$ be a nondecreasing, right continuous mapping and $T: X \to X$ be a mapping. T is called a Φ-max-contraction iff for all $x, y \in X$, $d(Tx, Ty) \leq \Phi(M(x, y))$.

The following result due to Daneš [3] is a special case of the above theorem.

Corollary. *Let $T: X \to X$ be a Φ-max-contraction. If for some $x \in X$,*

(a) $\delta(0(x)) < \infty$

(b) $\mathrm{cl}(0(x))$ *is complete,*

Then T has a unique fixed point in X.

Proof. It suffices to show that T is a max-contraction on X. Let $\varepsilon > 0$. Since $\Phi(\varepsilon) < \varepsilon$, choose any ε_0 such that $\Phi(\varepsilon) \leq \varepsilon_0 < \varepsilon$. Now, Φ being right continuous, therefore, there exists a $\delta_0 > 0$ such that $\Phi(t) \leq \varepsilon_0$ for all $t \in [\varepsilon, \varepsilon + \delta_0]$. We consider two cases. Case I. If $\varepsilon \leq M(x, y) < \varepsilon + \delta_0$, then since T is a Φ-max-contraction, therefore, $d(Tx, Ty) \leq \Phi(M(x, y)) \leq \varepsilon_0$. Case II. If $M(x, y) \leq \varepsilon$ then since Φ is nondecreasing, $\Phi(M(x, y)) \leq \Phi(\varepsilon) \leq \varepsilon_0$. In either case, the pair ε_0, $\delta_0 > 0$ satisfy (2) on X.

It may be pointed out that a recent result of Ćirić [2] is a special case of Daneš [3] and consequently of our above theorem.

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