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ON THE COMPLETION OF INCOMPLETE LATIN SQUARES

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1. A *latin rectangle* of order r by s based upon integers $1, 2, \ldots, n$ is defined as an array of r rows and s columns formed from the integers $1, 2, \ldots, n$ in such a way that the integers in each column are distinct. A latin rectangle of order r by s (of size $r \times s$) is called *incomplete* or *partial* if less than rs of its cells are occupied. If $r=s=n$ then latin rectangle is called *latin square of order n. Incomplete* (or *partial*) *latin square* of order n is an n by n array such that in some subset of the n^2 cells of array each of the cells is occupied by an integer from the set $\{1, 2, \ldots, n\}$ and that no integer from that set occurs in any row or column more than once. Thus, Fig. 1.1 is a latin square of order five, and Fig. 1.2 is an incomplete latin square of order five with six occupied cells.

5	1	2	4	3
2	4	3	5	1
1	2	4	3	5
3	5	1	2	4
4	3	5	1	2

Figure 1.1

1				
		2		1
5				
	3			
			5	

Figure 1.2

The question whether an incomplete latin square of order n can be completed to a latin square of order n, i. e. whether there exists a latin square of order n which agrees with the given latin square, is answered, in general, negatively, as Fig. 1.3 shows. More generally, for any $n>1$ there exist incomplete latin squares of order n with n occupied cells that cannot be completed.

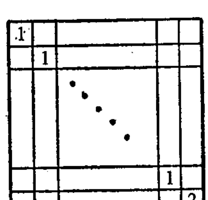

Figure 1.3

However, there is no example of an incomplete latin square of order n with less than n occupied cells that cannot be completed.

Evans ([1]) conjectured that every incomplete latin square of order n with $n-1$ or fewer occupied cells can be completed.

Several partial proofs of Evans conjecture are known (see [3]), but conjecture at present remains still open.

In this note we study the completion of incomplete latin squares on the basis of Evans conjecture.

Applied procedure is based upon the notion of *embedding*. An r by s latin rectangle is said to be embedded in a latin square of order n provided that it is possible to adjoin $n-s$ columns and $n-r$ rows in such a way that the resulting array is a latin square of order n.

In problems of embedding latin rectangles into latin squares the following theorem of Ryser is fundamental (see [4]).

Theorem 1.1. *Let T be an r by s latin rectangle based on the integers $1, 2, \ldots, n$. Let $N(i)$ denote the number of times that the integer i occurs in T. A necessary and sufficient condition in order that T may be embedded into an n by n latin square is that for each $i=1, 2, \ldots, n$*

$$N(i) \geqslant r+s-n.$$

Ryser's theorem is a generalisation of the following theorem of M. Hall (see [2]).

Theorem 1.2. *Any r by n latin rectangle may be extended to an n by n latin square.*

2. Theorem 2.1. *Given an incomplete latin square of order $n=2m-1$ with fewer than n occupied cells, if there is a row and a column such that there are at least $m-1$ occupied cells on both of them, then the square may be completed.*

Proof. If all occupied cells are on a line we easily complete that line and 1 by n latin rectangle embed in n by n latin square applying theorem of M. Hall.

In the sequel we exclude this case.

Now we establish two lemmas that may have some interest in their own right.

Lemma 2.1. *Any given line can be completed in an incomplete latin square of order n with fewer than n occupied cells.*

Proof. Through row and column permutations, and possibly transposition, we may assume that the given line is the first row and that the number of occupied cells on that line is $k \geqslant 0$. Denote by m_i the number of occupied cells in the i-th column off the first row. Then

$$m_1+m_2+\cdots+m_n+k \leqslant n-1. \tag{1}$$

By column permutations we may assume that all occupied cells off the first row are in tre first s column, that is $m_1 \cdot m_2 \cdot \cdots \cdot m_s > 0$ and $m_{s+1} = \cdots = m_n = 0$. Because of (1) $s < n$.

To fill in the cell (l, j) where $m_j > 0$, we have to avoid: k letters already existing on the first row, p letters filled in till the moment and m_j letters in the j-th column. But

$$p \leqslant m_{i_1} + m_{i_2} + \cdots + m_{i_p}$$

and we have to avoid

$$k + m_j + p \leqslant k + m_j + m_{i_1} + m_{i_2} + \cdots + m_{i_p}$$
$$\leqslant k + m_1 + m_2 + \cdots + m_n \leqslant n - 1.$$

Thus, the cell (l, j) can be filled in. When we fill in all the cells (l, j) with $m_j > 0$, the rest of the first row is easy to complete, since the only letters to avoid are letters in the first row.

Lemma 2.2 *In an incomplete latin square of order n with fewer than* $n-1$ *occupied cells off a given row (or column), we may complete any column (or row) exepting possibly the cell of intersection of those two lines.*

Proof. Without loss of generality we can assume that the given line is the first row and we want to complete the first column.

Denote by m_i the number of occupied cells in the i-th row off the first column, and with k the number of occupied cells in the first column off the first row. Then

(2) $$m_2 + m_3 + \cdots + m_n + k \leqslant n - 2.$$

Through row permutations we may assume that the first s rows contain all occupied cells off the first column, that is

(3) $$m_2 \cdot m_3 \cdot \cdots \cdot m_s > 0, \quad \text{and} \quad m_{s+1} = \cdots = m_n = 0.$$

(That $s < n$ it follows from (2).)

If the cell (2, 1) is empty we can immediately fill it in, since there are only $m_2 + k + 1 \leqslant n - 1$ letters to avoid.

Suppose the cells (2, 1), ..., $(s-1, 1)$ are filled in. To fill in the cell $(s, 1)$ we have to avoid: at most $k+1$ already existing on the first column, at most $s-2$ filled in till the moment and m_s letters on the s-th row. But because of (3), $s - 2 \leqslant m_2 + \cdots + m_{s-2}$, and, according to (2), we have to avoid at most

$$k + 1 + s - 2 + m_s \leqslant k + 1 + m_2 + \cdots + m_s + \cdots + m_n \leqslant n - 1$$

letters.

So the cell $(s, 1)$ can be filled in. In the rest of the first column we have to avoid only the letters already used in the column.

Remarks 1. If we exclude the supposition concerning the intersection cell of the two lines, we can obtain the latin square of Fig. 2.1. in which the cell $(1, n)$ of the last column cannot be filled (but all other cells of the last column can be filled).

2. The excluded cell can be occupied as well, in which case the prescribed line can be completed entirely.

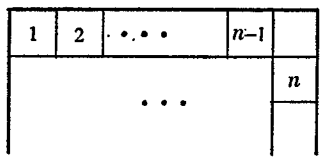

Fig. 2.1

We shall now prove the theorem.

By row and column permutations, we may assume that the two prescribed lines are the first row and the first column, and that the $n-1$ remaining occupied cells are in the m by m incomplete latin square L of Fig. 2.1.

We want to complete the first row, the first column and the rest of the incomplete latin square L, and then embed L in an n by n latin square, applying the theorem of Ryser.

Note that the completion of the two lines is possible according to lemmas 2.1 and 2.2, and that the rest of L can be completed, since to fill any empty cell of L we have at most $m-1+m-1=2m-2$ letters to avoid and we desposed with $2m-1$ letters.

In filling the empty cells of the two lines and the square L we want to insure the possibility of application of Rysers theorem. In other words, for any $i \in \{1, 2, \dots, 2m-1\}$ we must ensure

$$N(i) \geqslant m+m-(2m-1)=1.$$

In that aim we proceed as follows.

Complete at the beginning that line (of the given two lines) which contains more occupied cells (if both have the same number, than any). According to lemma 2.1 this can be done. In this way at least m letters are used in L.

If both lines have the same number of occupied cells and, after completion of one of them, no empty cell remains in the intersection of other line and the square L, then all occupied cells are on the two lines, and there are $m^2-(m+m-1)=(m-1)^2$ empty cells in L. In those cells we put at most $m-1$ letters not already used in L.

If on the other line (after the completion of one of them) there exists an empty cell, belonging also to L, we fill it with a letter not used in the intersection of completed line and in the square L. If it is possible, then L contains at least $m+1$ letters. If not, then at least $m+1$ letters are used in L, as the line with empty cell contains at most $m-2$ letters.

In all cases, after completing the two lines, there are at least $m+1$ letters used in L.

Let, after that, in L k letters do not occur $(1 \leqslant k \leqslant m-2)$.

At least one line of L must contain only one letter off the completed line which intersects it, or, otherwise, the square L can be reducible to an incomplete latin rectangle of size m by $m-1$ (or $m-1$ by m) for which the Ryser's theorem is immediately applicable (of course $m+(m-1)-(2m-1)=0$).

Complete that line of L with remaining k letters, which is possible, since there are $m-2$ empty cells and $k \leqslant m-2$.

Doing that, the conditions of Ryser's theorem are fulfilled.

After embedding the latin square L in a latin square of order n, we permute rows and columns to obtain the latin square which agrees with the given incomplete latin square.

The theorem is proved.

In the same manner (even more easily) we can prove the following theorem.

Theorem 2.2. *Given an incomplete latin square of order $n=2m$ with fewer than n occupied cells, if there is a row and a column such that at least m occupied cells are on both of them, then the sequare may be completed.*

Remark. In [5] A. L. Wells proved that a partial latin square can be completed if at least $m-2$ occupied cells are on a given line (Wells theorem contains corresponding formulation for $n=2m$). Our theorem is a generalisation of the theorem of Wells for the case if one of the two lines contains at least $m-2$ occupied cells, and some other intersecting line at least one occupied cell. Besides, our proofs are quite different.

3. A question. Let R be an incomplete latin square of order n with $r\,(r<n)$ completed rows and there are $n-1-r$ occupied cells beside these rows. Can R be completed?

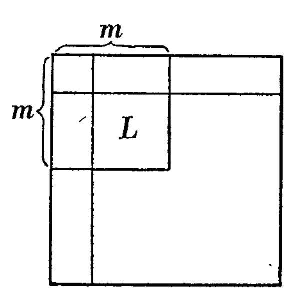

Fig. 2.2

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