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GRAPH EQUATIONS FOR LINE GRAPHS AND N-TH POWER GRAPHS I

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We shall consider only finite, undirected graphs without loops or multiple lines, or shortly, according to Harary [1], only graphs. For arbitrary graph G, $\overline{G}$ denotes a complement of G, $L(G)$ denotes a line graph of G and G^n denotes an n-th power graph of G, i. e. G^n is the graph having the same vertex set as G and in which two (different) vertices are adjacent if the distance between corresponding vertices in G is at most n. For all basic definitions and notations the reader is referred to [1], and for notion of graph equations, see for example, [2] or [3].

In this paper we shall solve the following three graph equations

(1) $$L(G)=G^n,$$

(2) $$\overline{L(G)}=G^n \text{ (or } L(G)=\overline{G^n}),$$

(3) $$L(G)=(\overline{G})^n.$$

In the above expressions, the equality sign means the isomorphism between corresponding graphs.

Note, that for $n=1$, the equation (1) is reduced to the wellknown result of V. V. Menon [4], while the equations (2) and (3) are reduced to the equation which is solved by M. Aigner [5]. So we shall assume that $n \geqslant 2$.

Equation $L(G)=G^n$.

T h e o r e m 1. *For any $n \geqslant 2$, the solutions to the equation $L(G)=G^n$ are graphs $G=mK_3$ where m is an arbitrary positive integer.*

P r o o f. Let $c(H)$ denote the number of vertices in the largest clique of a graph H. Then the following relations can be checked:

(a) $$c(L(G))=\begin{cases} \Delta(G) & \Delta(G)\geqslant 3 \text{ or } \Delta(G)\leqslant 1 \\ 3 & \Delta(G)=2 \text{ and } K_3\subseteq G, \\ 2 & \Delta(G)=2 \text{ and } K_3 \not\subseteq G \end{cases}$$

(b) $$c(G^n)\geqslant \Delta(G)+1,$$

* The results of this paper were obtained approximately at the same time by all three authors and accordingly, a joint paper is made.

where Δ denotes the maximal vertex degree of a graph and $\subseteq$ denotes the relation "to be an induced subgraph of". Namely, the relation (a) follows directly from Whitney's theorem for line graphs, while (b) is obvious.

Now on the basis of direct comparison of graph invariants $(c(L(G)) = c(G^n))$ we arrive to the proof of theorem.

Equation $\overline{L(G)} = G^n$.

Theorem 2. *For* $n \geqslant 2$, $G = C_{2n+3}$ *is the only solution to the equation* $\overline{L(G)} = G^n$.

Proof. Suppose $G = (\bigcup_{i \in I} G_i) \cup m K_1$, where G_i is nontrivial and connected for each $i \in I$ and $m \geqslant 0$. By substituting the last expression in (2) we get $|I| = 1$, since otherwise $\overline{L(G)}$ is connected and G^n disconnected. So $G = G_1 \cup m K_1$ and let us take $m \neq 0$. If $G_1 = K_2$ it is easy to see that (2) does not hold. If G_1 has more than one line then $K_3 \cup K_1 \subseteq G^n$, but since $K_{1,3} \not\subseteq L(G)$ we get a contradiction. Hence, G must be connected and due to the fact that $\overline{L(G)}$ and G^n have the same number of vertices G is a unicyclic graph.

Now we can prove the following two properties for a graph G:

(c) $$r(G) > n;$$

(d) no four vertices v_o, v_i, v_j, v_k of G satisfy the relations

$$d(v_o. v_s) \geqslant n+1 \quad (s = i, j, k) \text{ and } d(v_s, v_t) \leqslant n \quad (s, t = i, j, k);$$

where r denotes the radius of a graph and $d(u, v)$ denotes the distance between corresponding vertices of some graph. The property (c) follows from the connectedness condition. Namely, if $r(G) \leqslant n$ then there exists at least one vertex in G^n which is adjacent to all other vertices. But then, we have that $\overline{G^n}$ is disconnected while $L(G)$ is connected. So (c) holds. In order to prove (d) it is sufficient to observe that $K_{1,3}$ is forbidden as an induced subgraph in line graphs. So (d) also holds.

By using (c) and (d), under the supposition that G is different from a cycle, we can prove the following additional condition for G:

(e) if G is not a cycle, then G contains as an induced subgraph a graph from Fig. 1, where u_1 (u_2) is at the maximal distance from the nearest vertex v_1 (v_2) of the cycle and also the following relations $d(u_1, v_1) = d(u_2, v_2) = n + 1 - \left[\frac{1}{2} g(G)\right]$ and $d(v_1, v_2) = \left[\frac{1}{2} g(G)\right]$ (g denotes the girth of a graph) hold.

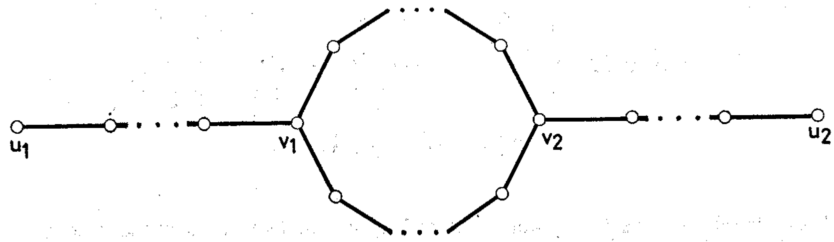

Fig 1.

For proving (e), suppose that u_1 is a vertex outside the cycle which is on the maximal distance from the nearest vertex v_1 of the cycle. We shall first show that $d(u_1, v_1) \leqslant n+1-\left[\frac{1}{2} g(G)\right]$. Suppose contrary. Then we can take u_1 and three consequtive vertices of the cycle, say w_1, w_2, w_3, for which $d(w_i, w_j) \leqslant n$ $(i, j=1, 2, 3)$ obviously hold and also for each i $d(u_1, w_i) \geqslant n+1$ since $d(u_1, w_i)=d(u_1, v_1)+d(v_1, w_i)$ and because we can choose w_1, w_2, w_3 such that $d(v_1, w_i) \geqslant \left[\frac{1}{2} g(G)\right]-1$ $(i=1, 2, 3)$ holds. But the last is in the contradiction with (d). As a consequence of the above just proved inequality we have $g(G) \leqslant 2n+1$. Now by using (c) it follows that in G there exists a vertex u_2 such that $d(v_1, u_2)>n$. Since $g(G) \leqslant 2n+1$ u_2 is outside the cycle and a vertex v_2 of the cycle such that $d(u_2, v_2)$ is minimal is different from v_1. Hence, we have just proved the existence of the graph from Fig. 1 in G. Now, since $n+1 \leqslant d(u_2, v_1)=d(u_2, v_2)+d(v_2, v_1)$, we have $n+1-d(v_1, v_2) \leqslant d(u_2, v_2) \leqslant n+1 - \left[\frac{1}{2} g(G)\right]$ and since $d(v_1, v_2) \leqslant \left[\frac{1}{2} g(G)\right]$ it follows that $d(u_2, v_2)=n+1- \left[\frac{1}{2} g(G)\right]$ and $d(v_1, v_2)=\left[\frac{1}{2} g(G)\right]$. On the basis of reciprocity $d(u_1, v_1) = d(u_2, v_2)$ and so (e) is proved.

Now assume that G is not a cycle and that $n+1-\left[\frac{1}{2} g(G)\right] \geqslant 2$ holds. In this case it is easy to see that (d) and (e) contradicts. To see this observe, for example, u_1, u_2, v_2 and any vertex of the (unique) path between u_2 and v_2 (note $d(u_2, v_2) \geqslant 2$). Thus we have $\left[\frac{1}{2} g(G)\right]=n$, i.e. $g(G)=2n+1$ or $g(G) = 2n$. For $g(G)=2n+1$, taking u_1, u_2, v_2 and one of the vertices of cycle which is adjacent to v_2 we get the same contradiction as above. Suppose now $g(G)=2n$. Due to (c) each vertex of the cycle must have at least one pendant line, while due to (d) each vertex of the cycle could have at most one pendant line (both facts can be easily verificated). So we have $G=C_{2n} \circ K_1$. But the last graph is not a solution since $L(G)$ and $\overline{G^n}$ have not the same degree sequences.

At last assume that G is a cycle, say $G=C_k$ for some k. Now $\overline{L(G)}$ is a regular graph of degree $k-3$ while G^n is a regular graph of degree $2n$ for $2n+1<k$ or a complete graph with k vertices for $2n+1 \geqslant k$. The only possibility for existing a solution is the case when $k=2n+3$. It can be easily shown that C_{2n+3} is a solution and this proves the theorem.

Equation $L(G)=(\overline{G})^n$.

Theorem 3. *The equation $L(G)=(\overline{G})^n$ has no solution for any $n \geqslant 2$.*

Proof. Suppose again that $G=(\bigcup_{i \in I} G_i) \cup m K_1$ where G_i is nontrivial and connected for each $i \in I$ and $m \geqslant 0$. If $m \neq 0$, $(\overline{G})^n$ is a complete graph, while $L(G)$ is complete only for $|I|=1$ and for G_1 being a star or a triangle. But then, $L(G)$ and $(G)^n$ are different since they have not the same number of vertices. If $m=0$, we immediately get $|I|=1$ and hence G must be connected. Since $L(G)$ and $(\overline{G})^n$ have the same number of vertices G is unicyclic.

For G being unicyclic we shall prove that $(\overline{G})^n$ is nearly always a complete graph. Suppose first that u and v are nonadjacent vertices in G. Then they are adjacent in $(\overline{G})^n$. The other possibility is that u and v are adjacent in G. If in G exists a vertex w nonadjacent to u and v then u and v are adjacent in $(\overline{G})^n$. So $(\overline{G})^n$ is always complete except for the case when in G exists a pair of adjacent vertices such that each of the remaining vertices is adjacent to at least one of them. But then G is a triangle such that only two of its vertices could have pendant lines or is a quadrangle such that only two of its adjacent vertices could have pendant lines. It is easy to see that C_3 and C_4 are not solutions. If only one vertex of the triangle has pendant lines then $(\overline{G})^n$ is disconnected while $L(G)$ is not. If just two vertices of the triangle have pendant lines then $(\overline{G})^n$ is a complete graph or for $n=2$ it contains K_5-x as an induced subgraph. Since $L(G)$ is complete only for G being a star or a triangle (isolated vertices are ignored) and since K_5-x is one of the Beineke's forbidden graphs for line graphs we have no solutions. If G can be obtained from a quadrangle by adding pendant lines to at most two adjacent vertices then $(\overline{G})^n$ is a complete graph or for $n=2$ and* $\overline{G}\neq C_4\cdot K_2$ it contains K_5-x as an induced subgraph. All above possibilities do not give any solution. Since $C_4\cdot K_2$ is not a solution the theorem is proved.

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* For dot-product see [1] p. 23.