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A FIXED POINT THEOREM FOR MAPPINGS WITH A SEQUENTIALLY COMPACT ITERATION IN PROBABILISTIC LOCALLY CONVEX SPACES

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In [2] a fixed point theorem for mapping $T: (S, \mathcal{F}, t) \to (S, \mathcal{F}, t)$ was proved, where $(S, \mathcal{F}, t)$ is a sequentially complete Hausdorff probabilistic locally convex space, t is a continuous t-norm [5] and, for every $i \in I$, the mapping T satisfies the following inequality:

$$(1) \qquad F^i_{Tx-Ty}(q(i)\varepsilon) \geqslant F^{f(i)}_{x-y}(\varepsilon), \text{ for every } (x, y, \varepsilon) \in S^2 \times R^+$$

where, for every $i \in I$:

$$(2) \qquad \overline{\lim_{n\to\infty}}\, q(f^n(i)) < 1$$

In this paper we shall prove a fixed point theorem for mapping $T: M \to M$ $(M \subset S)$ where $T^{n_0} M$ is sequentially compact subset of S in the (ε, λ)-topology and:

$$(3) \qquad \overline{\lim_{n\to\infty}}\, q(f^n(i)) = 1$$

If $f(i) = i$, for every $i \in I$ the relation (3) means that the mapping T is a non-expansive mapping and there are many fixed point theorems for such class of mappings if S is locally convex space ([1], [3], [4]).

In [3] the set $T^{n_0} M$ is compact, in [4] $\mathcal{L}$-densifying and in [1] Ψ-densifying.

Definition *Let S be a linear space over real or complex field K and for every i in the index set I is defined the function $\mathcal{F}^i: S \to \Delta^+$* (see [5]) *with the following properties ($\mathcal{F}^i(x)$ is denoted by F^i_x)*

1. *$F^i_o = H$, for every $i \in I$, where* $H(x) = \begin{cases} 0 & x \leqslant 0 \\ 1 & x > 0 \end{cases}$
2. *$F^i_{\mu x}(\varepsilon) = F^i_x\left(\frac{\varepsilon}{|\mu|}\right)$, for every $\mu \in K$, $\mu \neq 0$, every $x \in S$, every $\varepsilon > 0$, and every $i \in I$.*
3. *$F^i_{x+y}(\varepsilon_1 + \varepsilon_2) \geqslant t(F^i_x(\varepsilon_1), F^i_y(\varepsilon_2))$ for every $x, y \in S$, every $\varepsilon_1, \varepsilon_2 > 0$ and every $i \in I$ where $t: [0, 1] \times [0, 1] \to [0, 1]$ is t-norm* [5].

Then $(S, \mathcal{F}, t)$ is a probabilistic locally convex space.

The topology in S is introduced by the neighbourhood system of 0, $\mathcal{N}=\{N^i(\varepsilon, \lambda)\}_{(i, \varepsilon, \lambda)\in I\times R^+\times(0, 1)}$ where the set $N^i(\varepsilon, \lambda)$ is of the form:

$$N^i(\varepsilon, \lambda)=\{x \mid x\in S, F_x^i(\varepsilon)>1-\lambda\}$$

and in this topology S becomes a linear topological space if t is a continuous t-norm.

Further if $\{F_x^i=H, \text{ for every } i\in I\} \Leftrightarrow \{x=0\}$ then S is Hausdorff.

Theorem 1 [2] *Let $(S, \mathcal{F}, t)$ be a sequentially complete Hausdorff probabilistic locally convex space, where t be a continuous t-norm, and T be a mapping from S into S such that the inequalities* (1) *and* (2) *hold and that the following condition is satisfied:*

For every $(i, \lambda)\in I\times(0, 1)$ there exist $\varepsilon_{i,\lambda}\in R^+$ and $N_{i,\lambda}\in N$ such that for every $j\in\{f^s(i): s\geq N_{i,\lambda}\}$ $G_{x_0}^j(\varepsilon_{i,\lambda})>1-\lambda$ where:

$$G_{x_0}^i(\varepsilon)=\inf_{n\in N}\{F_{x_n-x_0}^i(\varepsilon)\}, \quad i\in I, \quad \varepsilon>0, \quad x_n=Tx_{n-1}, \quad n\in N$$

Then there exists one and only one element $x\in S$ such that:

(i) $x=Tx$

(ii) $\lim_{\varepsilon\to\infty} F_{x-x_0}^{f^s(i)}(\varepsilon)=1$ *for every $i\in I$, uniformly with respect to $s\in N$.*

For example, if $t=\min$ and there exists $x_0\in S$ such that for every $i\in I$, $\lim_{\varepsilon\to\infty} F_{Tx_0-x_0}^{f^n(i)}(\varepsilon)=1$, uniformly with respect to $n\in N$, and (1) and (2) hold, all the conditions of Theorem 1 are satisfied.

Theorem 2 *Let $(S, \mathcal{F}, t)$ be a sequentially complete Hausdorff probabilistic locally convex space where t is a continuous t-norm, M be a closed and star convex subset of S such that $\sup_{\varepsilon}\inf_{x, y\in M} F_{x-y}^i(\varepsilon)=1$, for every $i\in I$, T be a mapping from M into M such that* (1) *and* (3) *hold and that the following conditions are satisfied:*

1. *For every $i\in I$ there exist $g(i)\in I$ and $\Psi_i: R^+\to R^+$ such that:*

a) $\lim_{\varepsilon\to\infty}\Psi_i(\varepsilon)=\infty$

b) $F_x^{f^n(i)}(\varepsilon)\geq F_x^{g(i)}(\Psi_i(\varepsilon))$ *for every $\varepsilon>0$, $x\in S$, $n\in N$*

2. *$\overline{T^{n_0}M}$ is sequentially compact in (ε, λ)-topology. Then* Fix $(T)=\{x \mid x\in M, x=Tx\}\neq\varnothing$

Proof: As in [1] let $\{\lambda_n\}_{n\in N}$ be a sequence of real numbers from the interval (0, 1) and $\lim_{n\to\infty}\lambda_n=1$. For every $n\in N$ we shall define the mapping $T_n: M\to M$ in the following way:

$$T_n x=\lambda_n Tx+(1-\lambda_n)x_0$$

where x_0 is a star point from M. Since M is star convex it follows that $T_n M \subseteq M$, for every $n \in N$. Further if $Q_n(i) = \lambda_n q(i)$, then for every $i \in I$, $\varepsilon > 0$, $(x, y) \in M^2$ we have:

$$F^i_{T_n x - T_n y}(\varepsilon) = F^i_{\lambda_n Tx - \lambda_n Ty}(\varepsilon) = F^i_{Tx - Ty}\left(\frac{\varepsilon}{\lambda_n}\right) \geqslant F^{f(i)}_{x-y}\left(\frac{\varepsilon}{q(i)\lambda_n}\right) = F^{f(i)}_{x-y}\left(\frac{\varepsilon}{Q_n(i)}\right)$$

and so $\overline{\lim_{m\to\infty}} Q_n(f^m(i)) = \lambda_n \overline{\lim_{m\to\infty}} q(f^m(i)) = \lambda_n < 1$. Now, we shall prove that for every $n \in N$ the mapping T_n satisfies the last condition of Theorem 1. We have from the condition 1. of the Theorem:

$$G^{fr(i)}_{x_0, n}(\varepsilon) = \inf_{m \in N}\left\{F^{fr(i)}_{T^m_n x_0 - x_0}(\varepsilon)\right\} \geqslant \inf_{m \in N}\left\{F^{g(i)}_{T^m_n x_0 - x_0}(\Psi_i(\varepsilon))\right\} \geqslant \inf_{x, y \in M} F^{g(i)}_{x-y}(\Psi_i(\varepsilon)).$$

From the condition $\sup_{\varepsilon} \inf_{x, y \in M} F^i_{x-y}(\varepsilon) = 1$ and $\lim_{\varepsilon \to \infty} \Psi_i(\varepsilon) = \infty$ it follows that the condition: $G^j_{x_0, n}(\varepsilon_{i,\lambda}) > 1 - \lambda$, for every $j \in \{f^s(i) : s \in N\}$ is satisfied and that there exists one and only one element $x_n \in M$ such that:

$$x_n = T_n x_n = \lambda_n T x_n + (1 - \lambda_n) x_0$$

Now, we shall prove that from the condition $\sup_{\varepsilon} \inf_{x, y \in M} F^i_{x-y}(\varepsilon) = 1$ it follows that M is bounded in (ε, λ) topology.

Let V be a neighborhood of zero of the form:

$$V(i, \varepsilon, \lambda) = \{x \mid x \in S,\ F^i_x(\varepsilon) > 1 - \lambda\}$$

If there exists $\mu > 0$ such that:

$$\mu M \subset V \tag{4}$$

then M is bounded in (ε, λ)-topology. The relation (4) means that:

$$F^i_{\mu x}(\varepsilon) > 1 - \lambda, \text{ for every } x \in M \tag{5}$$

First, we shall prove that $\sup_{\varepsilon} \inf_{x \in M} F^i_x(\varepsilon) = 1$ for every $i \in I$. Namely, we have:

$$F^i_x(\varepsilon) \geqslant t\left(F^i_{x-x_0}\left(\frac{\varepsilon}{2}\right), F^i_{x_0}\left(\frac{\varepsilon}{2}\right)\right)$$

and so:

$$\inf_{x \in M} F^i_x(\varepsilon) \geqslant t\left(\inf_{x \in M} F^i_{x-x_0}\left(\frac{\varepsilon}{2}\right), F^i_{x_0}\left(\frac{\varepsilon}{2}\right)\right)$$

and since t is continuous we obtain:

$$\sup_{\varepsilon} \inf_{x \in M} F^i_x(\varepsilon) \geqslant t\left(\sup_{\varepsilon} \inf_{x \in M} F^i_{x-x_0}\left(\frac{\varepsilon}{2}\right), \sup_{\varepsilon} F^i_{x_0}\left(\frac{\varepsilon}{2}\right)\right) \geqslant$$

$$\geqslant t\left(\sup_{\varepsilon} \inf_{x, y \in M} F_{x-y}\left(\frac{\varepsilon}{2}\right), \sup_{\varepsilon} F^i_{x_0}\left(\frac{\varepsilon}{2}\right)\right) =$$

$$= t(1, 1) = 1$$

So, for every $\lambda\in(0, 1)$ and $i\in I$, there exist $\delta_{i,\lambda}>0$ such that:

$$\inf_{x\in M} F_x^i(\delta_{i,\lambda})>1-\lambda.$$

Then we have $F_x^i(\delta_{i,\lambda})>1-\lambda$ for every $x\in M$. *If* $\frac{\delta_{i,\lambda}}{\varepsilon}=\mu(i, \varepsilon, \lambda)$ we have $F_x^i(\mu(i, \varepsilon, \lambda)\varepsilon)>1-\lambda$ and so $F_x^i\left(\frac{\varepsilon}{\mu(i,\varepsilon,\lambda)}\right)>1-\lambda$ which means (4) Now, we have:

$$\lim_{n\to\infty} x_n-Tx_n=\lim_{n\to\infty}(\lambda_n-1)\,Tx_n+\lim_{n\to\infty}(1-\lambda_n)\,x_0=0 \tag{6}$$

because $TM\subseteq M$ and M is bounded in (ε, λ)-topology. Let us prove that from (6) it follows:

$$\lim_{n\to\infty} x_n-T^{n_0}x_n=0. \tag{7}$$

We have:

$$F^i_{x_n-T^{n_0}x_n}(\varepsilon)\geqslant t\left(F^i_{x_n-Tx_n+\,\cdots\cdots\,+T^{n_0-2}x_n-T^{n_0-1}x_n}\left(\frac{\varepsilon}{2}\right),\right.$$

$$\left.,F^i_{T^{n_0-1}x_n-T^{n_0}x_n}\left(\frac{\varepsilon}{2}\right)\right)\geqslant$$

$$\geqslant t\left(F^i_{Tx_n-x_n+\cdots T^{n_0-2}x_n-T^{n_0-1}x_n}\left(\frac{\varepsilon}{2}\right),\ F^{f^{n_0-1}(i)}_{x_n-Tx_n}\left(\frac{\varepsilon}{2\cdot\prod_{r=0}^{n_0-2}q(f^r(i))}\right)\right)\geqslant$$

$$\geqslant t\left(t\left(F^i_{x_n-Tx_n+\cdots\cdots T^{n_0-3}x_n-T^{n_0-2}x_n}\left(\frac{\varepsilon}{4}\right),\ F^{f^{n_0-2}(i)}_{x_n-Tx_n}\left(\frac{\varepsilon}{2^2\prod_{r=0}^{n_0-3}q(f^r(i))}\right)\right),\right.$$

$$\left.,F^{f^{n_0-1}(i)}_{x_n-Tx_n}\left(\frac{\varepsilon}{2\prod_{r=0}^{n_0-2}q(f^r(i))}\right)\right).$$

It is easy to prove that the following inequality holds:

$$F^i_{x_n-T^{n_0}x_n}(\varepsilon)\geqslant\underbrace{t\,(t\,(\,\cdots\cdots\,t}_{(n_0-1)\ \text{times}}\left(F^i_{x_n-Tx_n}\left(\frac{\varepsilon}{2^{n_0-1}}\right),\ F^{f(i)}_{x_n-Tx_n}\left(\frac{\varepsilon}{2^{n_0-1}q(i)}\right)\right),$$

$$,\cdots\cdots\cdots\cdots),\ F^{f^{n_0-2}(i)}_{x_n-Tx_n}\left(\frac{\varepsilon}{2^2\prod_{r=0}^{n_0-3}q(f^r(i))}\right)\Bigg),\ F^{f^{n_0-1}(i)}_{x_n-Tx_n}\left(\frac{\varepsilon}{2\cdot\prod_{r=0}^{n_0-2}q(f^r(i))}\right)\Bigg)$$

Let $\Phi(x_1, x_2, \ldots\ldots, x_{n_0})=t(t(\ldots\ldots(t(t(x_1, x_2), x_3), \ldots\ldots\ldots, x_{n_0})$ where $(x_1, x_2, \ldots\ldots, x_{n_0})\in[0, 1]^{n_0}$. Since t is a continuous mapping from $[0, 1]^2$ into $[0, 1]$ and $t(1, 1)=1$ it follows that Φ is a continuous mapping from

$[0, 1]^{n_0}$ into $[0, 1]$ and $\lim\limits_{(x_1, \ldots, x_{n_0}) \to (1, 1, \ldots, 1)} \Phi(x_1, x_2, \ldots, x_{n_0}) = \Phi(\underbrace{1, 1, \ldots\ldots, 1}_{n_0-\text{times}})$ $= \underbrace{t(t(\ldots\ldots t}_{(n_0-1)-\text{times}}(t(1, 1), 1), \ldots\ldots, 1) = 1$. So for a given $\delta \in (0, 1)$ there exists $\lambda_\delta \in (0, 1)$ such that:

$\Phi(x_1, x_2, \ldots\ldots, x_{n_0}) > 1 - \delta$ if $x_i > 1 - \lambda_\delta$, $i = 1, 2, \ldots\ldots, n_0$. From (6) it follows that there exists $N(i, \varepsilon, \lambda)$ such that:

$$F^i_{x_n - Tx_n}\left(\frac{\varepsilon}{2^{n_0-1}}\right) > 1 - \lambda, \quad F^{f^r(i)}_{x_n - Tx_n}\left(\frac{\varepsilon}{2^{n_0-r} \prod\limits_{s=0}^{r-1} q(f^s(i))}\right) > 1 - \lambda$$

$r = 1, 2, \ldots\ldots, n_0 - 1$, for every $n \geqslant N(i, \varepsilon, \lambda)$. So we have:

$$F^i_{x_n - T^{n_0}x_n}(\varepsilon) > 1 - \delta \text{ for every } n \geqslant N(i, \varepsilon, \lambda_\delta)$$

which means that the relation (7) is valid.

Since the mapping T is continuous and the set $\overline{T^{n_0}M}$ is sequentially compact there exists a subsequence $\{x_{n_k}\}_{k \in N}$ such that $\lim\limits_{k \to \infty} T^{n_0}x_{n_k} = y$ and $y = \lim\limits_{k \to \infty} x_{n_k} = \lim\limits_{k \to \infty} Tx_{n_k} = T(\lim\limits_{k \to \infty} x_{n_k}) = Ty$ because of (6) and (7). So $y \in Fix(T) \neq \varnothing$ and the proof is complete.

Corollary *Let S be a Banach space, M be a bounded, closed and convex subset of S and T be a mapping from M into M such that:*

1. $\| Tx - Ty \| \leqslant \| x - y \|$ *for every* $x, y \in M$.

2. *The set* $\overline{T^{n_0}M}$ *is compact.*

Then there exists $x \in M$ such that $x = Tx$.

Proof: It is known that S is a random normed space if:

$$F_x(\varepsilon) = \begin{cases} 1 & \|x\| < \varepsilon \\ 0 & \|x\| \geqslant \varepsilon \end{cases}$$

and the mapping t is min. Moreover the (ε, λ)-topology on S and the norm topology are the same. It is easy to see that from the condition 1. it follows that $F_{Tx-Ty}(\varepsilon) \geqslant F_{x-y}(\varepsilon)$ and that $\sup\limits_{\varepsilon} \inf\limits_{x, y \in M} F_{x-y}(\varepsilon) = 1$ since the set M is bounded. Here is $I = \{i\}$ and the mappings f, g and $\{\Psi(\varepsilon)\}$ are identical mapping.

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