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ON COMMON TRANSVERSALS OF SEPARATING FAMILIES

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It is given a brief proof of a theorem of Brown ([1]) on common transversals.

Let the symbol $\mathcal{F}$ denote a set of t families of sets, each of t families consisting of s finite but not necessarily distinct or nonempty sets. The symbol Ω will denote the union of all of the sets contained in all of the t families. Thus $\mathcal{F}=(F_1, F_2, \ldots F_t)$, where for each j, $1\leqslant j\leqslant t$, $F_j=(F_j(1), F_j(2), \ldots, F_j(s))$, $|F_j(i)|<\infty$ ($|F|$ denotes the cardinality of F), $\Omega=\cup\{F_j(i) \mid 1\leqslant j\leqslant t,\ 1\leqslant i\leqslant s\}$.

$\mathcal{F}$ is said to *separate points* of Ω if

$$|\cap\{F_j(a_j) \mid 1\leqslant j\leqslant t\}|\leqslant 1$$

for every t-iple $(a_1, a_2, \ldots, a_t)$, where $0\leqslant a_j\leqslant s$, $1\leqslant j\leqslant t$ and $F_j(0)=\Omega\setminus\cup\{F_j(i) \mid 1\leqslant i\leqslant s\}$.

A set T is a *transversal* of the family F_j if there is a bijection $\varphi: T\rightarrow\{1, 2, \ldots, s\}$ such that, for all $x\in T$, $x\in F_j(\varphi(x))$. The set T is *common transfersal* of $F_1, F_2, \ldots, F_t$ if T is simultaneously a transversal of each F_j, $1\leqslant j\leqslant t$.

In [1] T. S. Brown proved that if $\mathcal{F}$ is separating and satisfying some cardinality conditions, then there exists a common transversal of $\mathcal{F}$. He considered two cases.

Case 1. Each F_j covers Ω, that is $\Omega=\cup\{F_j(i) \mid 1\leqslant i\leqslant s\}$, and $|\Omega|>s^t-s^{t-1}$.

Case 2. For some $j\in\{1, \ldots, t\}$ $\cup\{F_j(i) \mid 1\leqslant i\leqslant s\}\neq\Omega$ and $|\Omega|>(s+1)^t-s^{t-1}-1$.

In both cases $\mathcal{F}$ has a common transversal.

Case 2 is readily reduced to the case 1, so we deal only with the case 1, proving the following

THEOREM. *Let* $\mathcal{F}$ *separate points and for each* $j\in\{1, \dots t\}$ $\Omega=\cup\{F_j(i)\mid 1\leqslant \leqslant i\leqslant s\}$. *If*

(C) $$|\Omega|>s^t-s^{t-1}$$

then $\mathcal{F}$ *has a common transversal.*

Proof. Let $X=F_1\times F_2\times\cdots\times F_t$. If $x\in X$, then $x=(F_1(i_1), F_2(i_2), \dots, F_t(i_t))$, $1\leqslant i_j\leqslant s$, $F_j(i_j)$ $(1\leqslant j\leqslant t)$ being coordinates of x. Since $\mathcal{F}$ separates points, we have $|\Omega|\leqslant|X|=s^t$. Denote by π a disjoint partition of X having the following two properties: (i) for every $P\in\pi$ is $|P|=s$; (ii) given $P\in\pi$ then for any $j\in\{1, \dots, t\}$ and every $i\in\{1, \dots, s\}$ there exists $x\in P$ such that $F_j(i)$ is a coordinate of x. Existence of such partitions is easilly proved by induction on t. Denote by Π the set of all such partitions. Let $\pi\in\Pi$, $P\in\pi$, and $x\in P$; we shall write

$$i(x)=\cap\{F_j(i_j)\mid 1\leqslant j\leqslant t\},$$

$$b(P)=\cup\{i(x)\mid x\in P\}.$$

Evidently if for some $\pi\in\Pi$ and some $P\in\pi$ $|b(P)|=s$, $\mathcal{F}$ has common transversal. Suppose $\mathcal{F}$ has no common transversal. Then for every $\pi\in\Pi$ and every $P\in\pi$ is $|b(P)|\leqslant s-1$. It follows

$$|\Omega|\leqslant\sum_{P\in\pi}|b(P)|\leqslant|\pi|\max|b(P)|\leqslant s^{t-1}(s-1)=s^t-s^{t-1},$$

contrary to the hypothesis.

REFERENCE

[1] T. S. Brown, *Common Transversals*, J. Combinatorial Theory (A) 21, 80—85 (1976).