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LOCALLY COMPACT SPACES $\mathcal{C}$-TAME AT INFINITY

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Abstract. L. C. Siebenmann's concept of a tame at ∞ locally compact (non-compact) space is generalized defining $\mathcal{C}$-tame at ∞ spaces, where $\mathcal{C}$ is an arbitrary class of topological spaces. The n-dimensional version of it, called n-tameness at ∞, is shown to be related to the fundamental dimension of compact metric spaces through the following complement theorem.

Theorem. *A Z-set X in the Hilbert cube Q has the fundamental dimension $\leqslant n$ if and only if Q-X is n-tame at ∞.*

1. Introduction. This paper is a continuation of our study of homotopy properties at infinity of locally compact (non-compact) spaces began in [4]. The key idea is the same only the property considered is different. Here we investigate spaces $\mathcal{C}$-tame at ∞. The concept is motivated by and generalizes Siebenmann's notion of a tame at ∞ space introduced for the purposes of studying manifolds which admit boundaries in [11] (see also [6]). The n-dimensional version of it, n-tameness at ∞, is shown to be closely related to the shape theoretic notion of fundamental dimension (Theorem (3.1)). That complement-type theorem is another example showing the close connection between shape theory and homotopy theory at ∞ of locally compact spaces formulated in [4].

The brief description of the content of the sections follows.

In § 2 we first define $\mathcal{C}$-tameness at ∞ for (non-compact) locally compact spaces. Then we prove several elementary theorems concerning it. In the formulations and in the methods of proofs they resemble corresponding theorems about $\mathcal{C}$-triviality at ∞ and $\mathcal{C}$-movability at ∞ from [4].

The main result of the paper is Theorem (3.1) in § 3 where we prove that a Z-set X in the Hilbert cube Q (or more generally in an arbitrary absolute neighborhood retract) has the fundamental dimension [2] $\leq n$ if and only if its complement $M = Q - X$ is n-tame at ∞, i. e., $\mathcal{P}^n$-tame at ∞ where $\mathcal{P}^n$ is the class of all finite polyhedra of dimension $\leq n$. This geometric characterization of the fundamental dimension can be used to get simplified

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proofs of results in [9]. We present only a short proof (Proposition (3.5)) of Nowak's estimate ([9]) $Fd(X_1 \cup X_2) \leq \max(Fd(X_1), Fd(X_2), Fd(X_1 \cap X_2)+1)$ of the fundamental dimension of the union of two compacta.

We assume that the reader is familiar with Borsuk's shape theory for compact metric spaces (see his recent book [2]). Only in (2.5) we shall use shape theory for arbitrary topological spaces in the form described by Kozlowski in [8]. A number of arguments use standard theorems and concepts of infinite dimensional topology. We recommend the survey [5] as a good source of information on this topic.

We keep notation from [4] and all undefined terms are taken from there. In particular, recall that all locally compact spaces (discriminately denoted M and N) are assumed non-compact. We use $\mathcal{C}$ to denote a fixed, but otherwise unless explicitely stated completely arbitrary, class of topological spaces. $\mathcal{C}^n$ consists of all members of $\mathcal{C}$ whose (covering) dimension is $\leq n$. $\mathcal{P}$ and $\mathcal{CW}$ denote the class of all finite CW-complexes and the class of all CW-complexes, respectively.

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2. *$\mathcal{C}$-tameness at ∞.* In the study of questions concerning the possibility of putting boundaries on finite-dimensional manifolds ([11]) and Q-manifolds ([6]) the concept of a tame at ∞ non-compact locally compact space has been useful. Here we shall introduce, in analogy with [4], a generalization of it named "$\mathcal{C}$-tame at ∞", where $\mathcal{C}$ is an arbitrary class of topological spaces. In this section we shall prove several elementary theorems concerning the property of $\mathcal{C}$-tameness at ∞ which do not require too many assumptions about $\mathcal{C}$.

Let $\mathcal{C}$ be a class of topological spaces. A locally compact non-compact space M is *$\mathcal{C}$-tame at ∞* provided that for every compact subset A of M there is a compact $B \supset A$ such that the inclusion $M - B \subset\!\!\to M - A$ factors up to homotopy through some member X of C, i. e. there are maps $M - B \to X$ and $X \to M - A$ making the diagram

$$\begin{array}{ccc} M-B & \subset\!\!\longrightarrow & M-A \\ & \searrow \quad \nearrow & \\ & X & \end{array}$$

homotopy commutative. A space M is *tame* (*n-tame*) *at* ∞ if it is $\mathcal{P}-(\mathcal{P}^n-)$ tame at ∞.

The following three examples illustrate this concept.
The reader will be able to construct many more using theorems from this paper (especially (3.1)).

(2.1.) Example. Let $A \subset X$ be a Z-set in a compact ANR X. Then $M = X - A$ is tame at ∞.

Proof. By (2.4) below, it suffices to prove that $M\times Q=X\times Q-A\times Q$ is tame at ∞. As $X\times Q$ is homeomorphic to $P\times Q$, where P is a finite polyhedron [5], one easily sees that $A\times Q$ has arbitrary small closed neighborhoods of the form $P'\times Q_n$ with P' a compact subpolyhedron of $P\times I^n$, for some n, where Q is represented in the standard way as the countable infinite product $\prod_{i>0} I_i$ of the unit interval $I_i=[-1, 1]$ while $I^n=\prod_{i=1}^n I_i$ and $Q_n=\prod_{i>n} I_i$. Thus, complements of big compact subsets of $M\times Q$ are homotopy equivalent to $P'\times Q_n$, since $A\times Q$ is a Z-set in $X\times Q$.

(2.2) Example. Let $\sigma=\{X_i, f_i\}_{i>0}$ be an inverse sequence of finite polyhedra and let Map(σ) be its infinite mapping cylinder [6]. Then Map(σ) is $\{X_i\}_{i>0}$-tame at ∞.

Proof. The proof is clear once it is known that Map(σ) is homotopy equivalent to X_1 [6].

(2.3) Example. A connected, locally connnected, and locally compact space M is $\mathcal{C}$-trivial at ∞, for every class $\mathcal{C}$, if and only if M is $\mathcal{D}_f$-tame at ∞, where $\mathcal{D}_f$ is the family of all finite discrete sets (see [3]).

We shall prove first that $\mathcal{C}$-tameness at ∞ is preserved under the relation of quasi-domination at ∞ introduced in [4]. Recall that a locally compact space M *quasi dominates at* ∞ another such space N provided that for every compact subset A of N there is a compact $B\supset A$ and proper maps $f: N\to M$ and $g: M\to N$ such that $g\circ f|_{N-B}$ is in $N-A$ homotopic to the inclusion $N-B\hookrightarrow N-A$.

(2.4) Theorem. *If a space M is $\mathcal{C}$-tame at ∞ and quasi-dominates at ∞ a space N, then N is also $\mathcal{C}$-tame at ∞.*

Proof. Let $A\subset N$ be an arbitrary compact subset. Since M quasi-dominates at ∞ N, there is $B_1\supset A$ and proper maps f: $N\to M$ and g: $M\to N$ such that $g\circ f|_{N-B_1}$ is in $N-A$ homotopic to the inclusion $N-B_1\hookrightarrow N-A$. Now, $g^{-1}(B_1)$ is a compact subset of M. The assumption that M is $\mathcal{C}$-tame at ∞ gives us a compact subset B' of M, a member X of $\mathcal{C}$, and maps $\alpha': M-B'\to X$ and $\beta': X\to M-g^{-1}(B_1)$ with the property that $\beta'\circ\alpha'$ is in $M-g^{-1}(B_1)$ homotopic to the inclusion $M-B'\hookrightarrow M-g^{-1}(B_1)$. Pick a compact subset B of N such that $f(N-B)\subset M-B'$ and define α: $N-B\to X$ and β: $X\to N-A$ as compositions $\alpha'\circ f|_{N-B}$ and $g\circ\beta'$, respectively. It is easy to see that $\beta\circ\alpha$ is in $N-A$ homotopic to the inclusion $N-B\hookrightarrow N-A$, i.e. that N is $\mathcal{C}$-tame at ∞.

For a locally compact space M with a sufficiently nice local structure the question whether M is $\mathcal{C}$-tame at ∞, where $\mathcal{C}$ is a class of CW-complexes, depends only on homotopy types of spaces in $\mathcal{C}$. To prove a slightly stronger form of that statement we shall use the shape theory of arbitrary topological spaces in the form described by Kozlowski [8].

A class of topological spaces $\mathcal{C}$ *shape dominates* a class $\mathcal{D}$ if for every $X\in\mathcal{D}$ there is $Y\in\mathcal{C}$ such that Y shape dominates X. In Kozlowski's approach this means that there are natural transformations $\mathcal{F}$: $[X, -]\to[Y, -]$ and $\mathcal{G}$: $[Y, -]\to[X, -]$ between functors $[X, -]$, $[Y, -]$: $\mathcal{H}\to Sets$, where $\mathcal{H}$ is the homotopy category of spaces having the homotopy type of CW-complexes, such that $\mathcal{G}\circ\mathcal{F}=\mathcal{I}d$.

(2.5) Theorem. *Let M be a locally compact ANR space (a locally compact locally n-connected metrizable space). Let $\mathcal{D}$ be a class of metrizable spaces (of dimension $\leq n$). If M is $\mathcal{D}$-tame at ∞, and a class $\mathcal{C}$ of CW-complexes shape dominates the class $\mathcal{D}$, then M is also $\mathcal{C}$-tame at ∞.*

Proof. We shall prove only the n-dimensional version. The same proof, slightly changed, applies to the case $M \in ANR$.

Let M be a locally compact locally n-connected metrizable space and let $A \subset M$ be a compact subset. By [7], there is a CW-complex P of dimension $\leq n$ together with a map Φ: $P \to M-A$ such that, for every map g: $X \to M-A$ defined on a metrizable space X with $\dim X \leq n$, there exists a map g^*: $X \to P$ such that g and $\Phi \circ g^*$ are homotopic (in $M-A$). Select $B \supset A$, $X \in \mathcal{D}$ and maps f: $M-B \to X$ and g: $X \to M-A$ such that the diagram

$$\begin{array}{ccc} M-B & \overset{i}{\hookrightarrow} & M-A \\ & {\scriptstyle f} \searrow \quad \nearrow {\scriptstyle g} & \\ & X & \end{array}$$

homotopy commutes. Since $\mathcal{D}$ is shape dominated by $\mathcal{C}$, there is $Y \in \mathcal{C}$ and natural transformation $\mathcal{F}$ and $\mathcal{G}$ as above. As Y is a CW-complex, there is a map a: $X \to Y$ that induces $\mathcal{G}$, i. e., $a^* = \mathcal{G}$. Let f': $M-B \to Y$ be the composition $a \circ f$ and let g': $Y \to M-A$ be the composition $\Phi \circ g^{**}$, where g^{**}: $Y \to P$ is a representative of the homotopy class $\mathcal{F}_P([g^*])$. The chain $[\Phi \circ g^{**} a \circ f] = \Phi_{\#} \circ f^{\#} \circ a^{\#}([g^{**}]) = \Phi_{\#} \circ f^{\#} \circ \mathcal{G}_P \circ \mathcal{F}_P([g^*]) = \Phi_{\#} \circ f^{\#}([g^*]) = [\Phi \circ g^* f] = [g \circ f] = [i]$ shows that $g' \circ f' \simeq i$. Hence. M is $\mathcal{C}$-tame at ∞.

As there are only countably many homotopy types among compact polyhedra (of dimension $\leq n$) [1], we immediately get

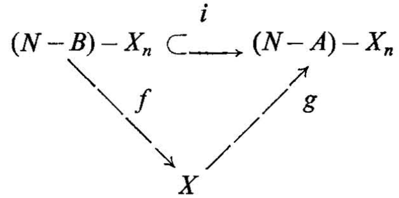
(2.6) Corollary. *There is a sequence $P_1, P_2 \ldots$ of finite polyhedra (of dimension $\leq n$) such that an ANR (an LC^n metrizable space) M is tame (n-tame) at ∞ if and only if M is $\{P_1, P_2, \ldots\}$-tame at ∞.*

Our next theorem in view of the example (2.3) can be considered as an impovement of (3.8) in [4]. The assumption of $\mathcal{C}$-unstability there is weakened here to the requirement of one-sided global unstability that we now define.

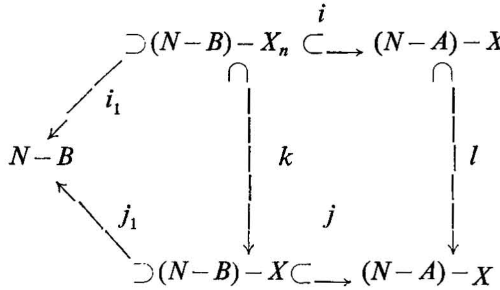
A closed subset A of a space X is called *globally left (right) unstable in* X if for each open neighborhood U of A in X the inclusion $U-A \hookrightarrow U$ has a left (right) homotopy inverse.

(2.7) Theorem. *Let N be a compact space and let $X_1 \supset X_2 \supset \ldots$ be a decreasing sequence of its closed subsets. Suppose each X_i is globally right unstable in N and $X = \bigcap_{i>0} X_i$ is globally left unstable in N. If the complements $M_i = N - X_i$ are $\mathcal{C}$-tame at ∞, then $M = N - X$ is $\mathcal{C}$-tame at ∞.*

Proof, Let $A \subset M$ be a compact subset. The set $N-A$ is an open neighborhood of X in N. Since X is the intersection of X_i's we can find $n \geq 1$ such that $N-A$ is an open neighborhood of X_n. But $M_n = N - X_n$ is $\mathcal{C}$-tame at ∞, so that there is a compact $B \subset M_n$, a member X of $\mathcal{C}$, and maps f: $M_n - B = (N-B) - X_n \to X$ and g: $X \to M_n - A = (N-A) - X_n$ making the diagram



homotopy commutative. In the commutative diagram of inclusions

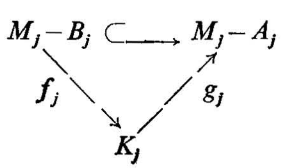


i_1 has a right homotopy inverse i_1^R and j_1 has a left homotopy inverse j_1^L, Then $k \simeq j_1^L i_1$ has a right homotopy inverse $k^R = i_1^R \circ j_1$. Hence, $j \simeq l \circ i \circ k^R$. Finally, j is homotopic to $(l \circ g) \circ (f \circ k^R)$ proving that M is $\mathcal{C}$-tame at ∞.

The product of spaces $\mathcal{C}$-tame at ∞ need not be $\mathcal{C}$-tame at ∞ (for example, the real line R is $\{X\}$-tame at ∞, where X is the subspace $\{0, 1\}$ of $[0, 1]$, while the plane $R^2 = R \times R$ is not). But we can prove the following result related to products.

(2.8) Theorem. *Let N_i be a compact contractible space and let $X_i \subset N_i$ be a closed subset, for each $i = 1, 2, 3, \ldots$.*
Put $N = \prod_{i=1}^{\infty} N_i$, $X = \prod_{i=1}^{\infty} X_i$, $M_i = N_i - X_i$, and $M = N - X$. If each M_i is $\mathcal{C}_i$-tame at ∞, each X_i is globally right unstable in N_i, and X is globally left unstable in N, then M is $\mathcal{C}_1 \times \mathcal{C}_2 \times \cdots$ -tame at ∞, where $C_1 \times C_2 \times \cdots$ denotes the class of all finite products $K_1 \times \cdots \times K_n$ with $K_i \in C_i$.

Proof. Let A be a compact subset of M. The set $N - A$ is an open neighborhood of X in N. Hence, there is $n \geqslant 1$ and there are compact subsets $A_1 \subset M_1, \ldots, A_n \subset M_n$ such that $X \subset (N_1 - A_1) \times \cdots\cdots \times (N_n - A_n) \times \prod_{k>n} N_k$. Since each M_i is $\mathcal{C}_i$-tame at ∞ we can find compact subsets B_j of M_j, spaces $K_j \in \mathcal{C}_j$, and maps $f_j : M_j - B_j \to K_j$ and $g_j : K_j \to M_j - A_j$ such that the diagram



homotopy commutes, $j=1, \ldots, n$. Let $B=M-(N_1-B_1)\times \cdots \times (N_n-B_n)\times \times \prod_{k>n} N_k$. In the ladder

$$\begin{array}{c}
M-B=(N_1-B_1)\times \cdots \times (N_n-B_n)\times \prod_{k>n} N_k - X \\
\downarrow \cap \\
(N_1-B_1)\times \cdots \times (N_n-B_n)\times \prod_{k>n} N_k \\
\downarrow j_1^R \times \cdots \times j_n^R \times id \\
(M_1-B_1)\times \cdots \times (M_n-A_n)\times \prod_{k>n} N_k \\
\downarrow \text{proj} \\
(M_1-B_1)\times \cdots \times (M_n-B_n) \\
\downarrow f_1 \times \cdots \times f_n \\
K_1 \times \cdots \times K_n \\
\downarrow g_1 \times \cdots \times g_n \\
(M_1-A_1)\times \cdots \times (M_n-A_n) \\
\downarrow \times o \\
(M_1-A_1)\times \cdots \times (M_n-A_n)\times \prod_{k>n} N_k \\
\downarrow k_1 \times \cdots \times k_n \times id \\
(N_1-A_1)\times \cdots \times (N_n-A_n)\times \prod_{n>k} N_k \\
\downarrow l^L \\
(N_1-A_1)\times \cdots \times (N_n-A_n)\times \prod_{k>n} N_k - X \\
\downarrow \cap \\
M-A
\end{array}$$

j_i^R is a right homotopy inverse of the inclusion $j_i\colon M_i-B_i \hookrightarrow N_i-B_i$, proj is the projection, $\times 0$ is the obvious embedding, k_i is the inclusion $M_i-A_i \hookrightarrow N_i-A_i$, and l^L is a left homotopy inverse of the inclusion l: $(N_1-A_1)\times \cdots \times (N_n-A_n)\times \prod_{k>n} N_k - X \hookrightarrow (N_1-A_1)\times \cdots \times (N_n-A_n)\times \prod_{k>n} N_k$, $i=1, \ldots, n$. The composition of all maps in the above diagram is homotopic to the inclusion $M-B \hookrightarrow M-A$. Thus, M is $\mathcal{C}_1 \times \mathcal{C}_2 \times \cdots$-tame at ∞.

The theorem (3.13) in [4] can be improved to the following result. First recall that a non-compact locally compact space M is *$\mathcal{CW}$-trivial at ∞* if for every compact set A in M there is a larger compact set B such that every map of a CW-complex into a component of $M-B$ is null-homotopic in $M-A$.

(2.9) Theorem. *Let N be the union of compacta N_1 and N_2 intersecting in a compact ANR space N_0. Let $X \subset N$ be a closed connected subset such that $X_0=X\cap N_0$ is connected and such that $M_0=N_0-X_0$ is contractible and $\mathcal{CW}$-trivial at ∞ and $M_1=N_1-X$ and $M_2=N_2-X$ are one-ended. If $M=N-X$ is $\mathcal{C}$-tame at ∞, then both M_1 and M_2 are $\mathcal{C}$-tame at ∞.*

Proof. Consider an arbitrary compact subset $A_1 \subset M_1$. $A_0 = N_0 \cap A_1$ is a compact subset of M_0. One easily constructs a proper retraction of M_2 onto M_0 (see [10, Theorem (4.5)]) and, therefore, also a proper retraction r: $M \to M_1$. Hence, there is a compact subset A of M such that $r(M-A) \subset M_1 - A_1$. Since M is $\mathcal{C}$-tame at ∞, there is a compact $B \subset M$, a space $K \in \mathcal{C}$, and maps $f: M-B \to K$ and g: $K \to M-A$ making the diagram

$$\begin{array}{ccc} M-B & \hookrightarrow & M-A \\ & {\scriptstyle f} \searrow \quad \nearrow {\scriptstyle g} & \\ & K & \end{array}$$

homotopy commutative. Put $B_1 = B \cap M_1$. One easily checks that the diagram

$$\begin{array}{ccc} M_1 - B_1 & \hookrightarrow & M_1 - A_1 \\ \cap \downarrow & & \uparrow r \mid M-A \\ M-B & \hookrightarrow & M-A \\ & {\scriptstyle f} \searrow \quad \nearrow {\scriptstyle g} & \\ & K & \end{array}$$

homotopy commutes. Hence, M_1 is $\mathcal{C}$-tame at ∞. In a similar way we prove that M_2 is $\mathcal{C}$-tame at ∞.

An end e of a locally compact space M is *$\mathcal{C}$-tame*, where $\mathcal{C}$ is a class of topological spaces, if for every neighborhood U of e in FM, the Freudenthal compactification of M, there is another neighborhood $V \subset U$ of e, an $X \in \mathcal{C}$, and maps $f: V \cap M \to X$ and g: $X \to U \cap M$ such that the diagram

$$\begin{array}{ccc} V \cap M & \overset{i}{\hookrightarrow} & U \cap M \\ & {\scriptstyle f} \searrow \quad \nearrow {\scriptstyle g} & \\ & X & \end{array}$$

is homotopy commutative, i. e., the inclusion i is in $U \cap M$ homotopic to the composition $g \circ f$.

(2.10) Theorem. (*a*) *Let $\mathcal{C}$ be a component hereditary class of spaces closed under the formation of disjoint unions. If a locally compact space M is $\mathcal{C}$-tame at ∞, then each end e of M is $\mathcal{C}$-tame.*

(*b*) *Let a class $\mathcal{C}$ be closed under the formation of finite disjoint unions. If each end e of a locally compact space M is $\mathcal{C}$-tame, then M is $\mathcal{C}$-tame at ∞.*

Proof. (a) Let $e \in EM$ be an end of M and let U' be a neighborhood of e in FM. Select a neighborhood $U \subset U'$ of e such that $U \cap EM$ is both open and closed in EM, the end set of M. Then there are disjoint open sets $U_1 = U, U_2, U_3, \ldots, U_n$ of FM whose union covers EM. Let $A = M - \bigcup_{i=1}^{n} U_i$. Note that A is a compact subset of M. Since M is $\mathcal{C}$-tame at ∞, we can find a compact $B \supset A$, an $X \in \mathcal{C}$, and maps $f: M - B \to X$ and $g: X \to M - A$ making the diagram

$$\begin{array}{ccc} M-B & \hookrightarrow & M-A \\ & {}_{f}\searrow \quad \nearrow_{g} & \\ & X & \end{array}$$

homotopy commutative. Clearly, $g^{-1}(U \cap M) = Y$ is the union of components of X and $f((M-B) \cap U) \subset Y$. Put $V = U \cap ((M-B) \cup EM)$. The open set V is a neighborhood of e in FM and the diagram

$$\begin{array}{ccc} V \cap M & \hookrightarrow & U \cap M \\ & {}_{f|}\searrow \quad \nearrow_{g|} & \\ & Y & \end{array}$$

homotopy commutes. As $Y \in \mathcal{C}$ it follows that the end e is $\mathcal{C}$-tame.

(b) Let $A \subset M$ be a compact subset. $U = FM - A$ is an open set containing EM. Hence, we can find finitely many disjoint open sets $V_1, V_2, \ldots, V_n$ and $X_1 \in \mathcal{C}, \ldots, X_n \in \mathcal{C}$ such that $V = \bigcup_{i=1}^{n} V_i \supset EM$ and the inclusion $V_i \cap M \hookrightarrow U \cap M$ factors up to homotopy through X_i, for each $i = 1, \ldots, n$. It is easy to see that the inclusion $V \cap M \hookrightarrow U \cap M$ factors up to homotopy through the disjoint union X of X_i's. Put $B = M - \bigcup_{i=1}^{n} V_i$. Clearly, $M - B \hookrightarrow M - A$ homotopy factors through X. Hence, M is $\mathcal{C}$-tame at ∞, since $X \in \mathcal{C}$.

3. Fundamental dimension and n-tameness at ∞.

We shall get a characterization (Theorem (3.1)) of the fundamental dimension of a compact metric space X in terms of n-tameness at ∞ of its complement $M = Y - X$ in a compact ANR X containing X as a Z-set. Using this characterization of the fundamental dimension, we see that the result of the previous section imply (and, therefore, generalize) some theorems on the fundamental dimension from [9]. We believe that this is yet another piece of evidence that shape theory of compacta should be considered a part of the homotopy theory at ∞ of non-compact locally compact spaces [4].

The *fundamental dimension* $Fd(X)$ [2] of a compactum X is defined as $\min\{\dim Y \mid Y \text{ shape dominates } X\}$.

(3.1) Theorem. *Let Y be a compact ANR and let $X, X \subset Y$, be a Z-set in Y. Then $Fd(X) \leq n$ if and only if $M = Y - X$ is n-tame at ∞.*

Proof. As in [4, theorem (3.2)], without loss of generality we can assume $Y=Q$, the Hilbert cube.

Suppose X is a Z-set in Q and $Fd(X)\leq n$. Then there is an n-dimensional compactum Z such that $Sh(Z)\geq Sh(X)$. We can represent Z as the inverse limit $\varprojlim \sigma$ of an inverse sequence $\sigma=\{P_i, f_i\}$ where $P_1=$ point and $\dim P_i\leq n$, for every $i>0$. The product $\mathrm{Map}(\sigma)\times Q$, where $\mathrm{Map}(\sigma)$ is the infinite mapping cylinder of σ [6], when compactified by adding $(\varprojlim \sigma)\times Q$ is homeomorphic to Q and $(\varprojlim \sigma)\times Q$ is a Z-set in $(\mathrm{Map}(\sigma)\cup\varprojlim \sigma)\times Q\cong Q$ [6, Theorem 1.5]. Since $Sh(Z)=Sh(\varprojlim \sigma\times Q)\geq Sh(X)$, $\mathrm{Map}(\sigma)\times Q$ homotopy dominates at ∞ a space $M=Q-X$([4, Theorem 2.6)]). Hence if we prove that $\mathrm{Map}(\sigma)\times Q$ is n-tame at ∞ it will follow from Theorem (2.4) that M is n-tame at ∞.

Let $A\subset \mathrm{Map}(\sigma)\times Q$ be an arbitrary compactum. Pich k large enough so that $A\subset \mathrm{Map}(\{P_1\leftarrow P_2\leftarrow\cdots\leftarrow P_{k-1}\})\times Q=B$. We shall prove that there is a homotopy $g_t: V\to V$, with $V=\mathrm{Map}(\sigma)\times Q-B$, such that $g_0=id$ and $g_1(V)$ is a copy of P_k.

The proof of that is rather simple but the notation is cumbersome. Let $M_1=\mathrm{Map}(\{P_k\leftarrow P_{k+1}\leftarrow\cdots\})\times Q$, let $M_2=M_1\cup[(\varprojlim\{P_k\leftarrow P_{k+1}\leftarrow\cdots)\}\times\times Q]$, and let $M_3=P_k\times Q$. In the diagram

$$V \underset{d_1}{\overset{i_2}{\rightleftarrows}} M_1 \underset{\gamma}{\overset{i_1}{\rightleftarrows}} M_2 \xrightarrow{h} M_3$$

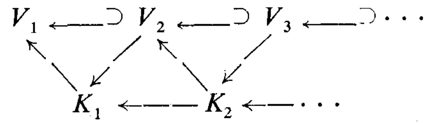
maps i_1 and i_2 are inclusions, d_1 is the end of the obvious strong deformation $d_t\,(0\leq t\leq 1)$ of V onto M_1 that slides $P_k\times[0,\ 1)\times Q$ onto $P_k\times\{0\}\times Q$, and γ is a homotopy inverse of i_1. It exists since $\varprojlim\{P_k\leftarrow P_{k+1}\leftarrow\cdots\}\times Q$ is a Z-set in the compactification M_2 of M_1. Let $e_t\,(0\leq t\leq 1)$ be a homotopy connecting id and $\gamma\circ i_1$. The map h is a homeomorphism of the mentioned compactification onto M_3 (its existence follows from [6, Theorem 1.5]). Finally, let λ_t, $0\leq t\leq 1$, denote the map mapping a point (x, q) in $M_3=P_k\times Q$ into $(x, (1-t)\,q)$.

Then our g_t is defined by

$$g_t=\begin{cases} d_{3t}, & 0\leq t\leq 1/3,\\ i_2\circ e_{3(t-1/3)}\circ d_1, & 1/3\leq t\leq 2/3,\\ i_2\circ\gamma\circ h^{-1}\circ\lambda_{3(t-2/3)}\circ h\circ i_1\circ d_1, & 2/3\leq t\leq 1.\end{cases}$$

Observe that by the Mapping Replacement Theorem [5] we can (up to homotopy) assume γ is an embedding so that $g_1(V)$ is indeed a copy of P_k.

Conversely, suppose a Z-set X in Q has the property that $M=Q-X$ is n-tame at ∞. Then we can find a sequence $M=V_1\supset M=V_2\supset V_3\supset V_4\supset\cdots$ of open subsets of M with compact complements and $\bigcap_{i>0} V_i=\varnothing$ such that for some sequence $K_1 = \text{point}$, K_2, $K_3 \dots$, where $K_2, K_2, \dots$ are n-dimensional finite polyhedra, we can form a homotopy commutative diagram



In the Appendix II of [6] it was shown that under these conditions there is a proper homotopy equivalence $\mathrm{Map}(\sigma) \times Q \to M$, where $\sigma = \{K_1 \leftarrow K_2 \leftarrow \cdots\}$. Since both M and $\mathrm{Map}(\sigma) \times Q$ are contractible Q-manifolds admitting boundaries (as defined in [6]) by [6, Theorems 7 and 9], $M \cong \mathrm{Map}(\sigma) \times Q$ and $Sh(\underleftarrow{\lim}\, \sigma \times Q) = Sh(\underleftarrow{\lim}\, \sigma) = Sh(X)$. But $\dim(\underleftarrow{\lim}\, \sigma) \leq n$ so that $Fd(X) \leq n$.

(3.2) Remark. It is clear from the above proof that a Z-set X in a Q-manifold Y has fundamental dimension $\leq n$ if and only if X has arbitrary small Q-manifold neighborhoods of the form $K \times Q$ where K is an at most n-dimensional finite complex. Also, had we assumed $Fd(X) = n$ then, with the notation from the second half of the proof for Theorem (3.1), $\dim(\underleftarrow{\lim}\, \sigma)$ must be equal n for otherwise M would be (n-1)-tame at ∞ and thus $Fd(X) \leq n - 1$, by the first half, which is a contradiction. Hence, we proved.

(3.3) Corollary. *If X is a compactum and $Fd(X) = n$, then there is an n-dimensional compactum Z such that $Sh(X) = Sh(Z)$.*

Corollary (3.3) was earlier proved by Holsztyński (unpublished) and Nowak [9].

(3.4) Corollary. *Let a compactum X quasi-dominates a compactum Y. If X has fundamental dimension $\leq n$, then Y also has fundamental dimension $\leq n$.*

Proof. Combine Theorems (3.1) and (2.4) in the present paper and Theorem (2.6) from [4].

The geometric characterization of the fundamental dimension achieved in Theorem (3.1) allows a simplification of some proofs from [9]. In this paper we give an example (Proposition (3.5)); others are included in the author's dissertation.

S. Nowak [9] gave a rather complicated proof of the next proposition. The proof we present shows some merits of our approach.

(3.5) Proposition. *Let X_1, X_2 be compacta. Then*

$$Fd(X_1 \cup X_2) \leqslant \max(Fd(X_1),\ Fd(X_2),\ Fd(X_1 \cap X_2) + 1).$$

Proof. Denote the integers appearing in the above inequality by k, l, m, n, respectively. Hence, we must prove $k \leq \max(l, m, n)$. Consider $X = X_1 \cup X_2$ as a Z-set in $Q \times [-1, 1]$ with $X_1 \subset Q \times [-1, 0]$, $X_2 \subset Q \times [0, l]$, and $X \cap (Q \times \{0\}) = X_0 = X_1 \cap X_2$ being a Z-set in $Q \times \{0\}$. Since $Fd(X_1) = l$, $Fd(X_2) = m$, there are (see Remark (3.2)) arbitrary small closed Q-manifold neighborhoods N_1 of X_1 in $Q \times [-1, 0]$ and N_2 of X_2 in $Q \times [0, 1]$ such that

$N_1 = K_1 \times Q$, $N_2 = K_2 \times Q$, where K_1 and K_2 are finite complexes of dimensions l and m, respectively. Pick a closed Q-manifold neighborhood $N_0 \subset N_1 \cap N_2$ of X_0 in $Q \times \{0\}$ that is homeomorphic to $K_0 \times Q$ for some (n-1)-dimensional finite complex K_0. But N_0 is a Z-set submanifold in both N_1 and N_2 so that, by Champan's relative triangulation theorem [5], we can find complexes and homeomorphisms $h_1 : N_1 \to K_1' \times Q$, $h_2 : N_2 \to K_2' \times Q$ extending a fixed homeomorphism $h_0 : N_0 \to K_0 \times Q$. Moreover, $\dim K_1' = \max(l, n)$ and $\dim K_2' = \max(m, n)$. Let $N \subset N_1 \cup N_2$ be any neighborhood of X for which $N \cap (Q \times \{0\}) = N_0$, and let K be the result of gluing K_1' and K_2' along K_0. We claim that the inclusion $N - X \subset_{\rightarrow} (N_1 \cup N_2) - X$ factors up to homotopy through K. To see this, let $\lambda_t : Q \times [-1, 1] \to Q \times [-1, 1]$ move $Q \times [-1, 1]$ off of X with $\lambda_t |_{(Q \times [-1, 1]) - N} = id$, for all t. Define $\alpha : N - X \to K$ by $\alpha = p \circ h_N |_{N-X}$, where $h_N = (h_1 |_{N \cap N_1}) \cup \cap h_2 |_{N \cap N_2})$ and p is the projection $K \times Q \to K$. Also $\beta : K \to (N_1 \cup N_2) - X$ is given by $= \lambda_1 \circ h_N^{-1} \circ (\times 0)$. One can easily chek that the diagram

$$\begin{array}{ccc} N - X & \hookrightarrow & (N_1 \cup N_2) - X \\ & \alpha \searrow \quad \nearrow \beta & \\ & K & \end{array}$$

homotopy commutes, which proves that $(Q \times [-1, 1]) - X$ is $\max(l, m, n)$-tame at ∞, i.e., $k \leq \max(l, m, n)$.

(3.6) Remark. If $X_0 = X_1 \cap X_2$ has trivial shape, then we can take K_0 to be a point and then $Fd(X_1 \cup X_2) = \max(Fd(X_1), (Fd(X_2))$ ([9, Theorem (4.19]).

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