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GENERALIZED STIRLING NUMBERS AND POLYNOMIALS

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1. Introduction

The infinite family of Stirling numbers (cf. [4], p. 34), which goes back to E. T. Bell (1939), did have some interest in number theory [1]. The Stirling numbers of the second kind and the corresponding polynomials [4] are defined by

(1.1) $$S(n, k)=\frac{(-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^n=\frac{1}{k!}\Delta^k O^n$$

and

(1.2) $$A_n(x)=\sum_{k=0}^{n}S(n, k)\,x^k,$$

respectively.

Recently, Singh [5], Sinha and Dhawan [6], and Shrivastava [7], studied the Stirling numbers and polynomials defined, respectively, by

(1.3) $$S^a(n, k, r)=\frac{(-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\alpha+rj)^n$$

and

(1.4) $$T_n^\alpha(x, r, -p)=\sum_{k=0}^{n}S^a(n, \mathrm{k}, r)\,p^r x^{rk}.$$

It is easily verified from (1.1) and (1.3) that

(1.5) $$S^a(n, k, r)=\sum_{i=0}^{n}\binom{n}{i}\alpha^{n-i}r^i S(i, k).$$

Thus the generalized Stirling numbers studied in the recent papers [5], [6] and [7] are merely linear combinations of the Stirling numbers of the second kind, defined by (1.1).

In connection with the study of the polynomials defined by (cf. [8], p. 75, Eq. (1.3); see also [2])

(1.6) $$T_n^{(\alpha, k)}(x, r, p)=x^{-a}\,e^{px^r}\,\Omega_x^n\{x^a e^{-px^r}\},\qquad \Omega_x\equiv x^k\,d/dx,$$

the author [3] introduced the following generalizations of the Stirling numbers and polynomials defined by (1.3) and (1.4), respectively

$$S^{(\alpha,k)}(n,m,r)=\frac{(-1)^m}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(\alpha+rj)^{[k-1,n]}, \tag{1.7}$$

$$T_n^{(\alpha,k)}(x,r,-p)=x^{n(k-1)}\sum_{m=0}^{n}S^{(\alpha,k)}(n,m,r)p^m x^{rm}, \tag{1.8}$$

where

$$\alpha^{[k-1,n]}=\left(\frac{\alpha}{k-1}\right)_n (k-1)^n, \tag{1.9}$$

α, $k\neq 1$, p, r are arbitrary complex numbers, and m, n are positive integers. Evidently, when $k\to 1$, equations (1.7) and (1.8) would reduce to (1.3) and (1.4), respectively, which, in turn, will yield (1.1) and (1.2), respectively, for $r-1=\alpha=0$.

The object of this paper is to apply certain operational techniques to study various properties of the generalized Stirling numbers and polynomials, defined by (1.7) and (1.8) above.

2. Generating Function

For the forward difference operator $\Delta_{\alpha,r}$, it is well known that [8, p. 77, Eq. (2.3)]

$$\Delta_{\alpha,r}^{i}f(\alpha)=\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}f(\alpha+rj), \tag{2.1}$$

which yields

$$\Delta_{\alpha,r}^{i}(\alpha+rj)^{[k-1,n]}=i!\sum_{m=0}^{n}\binom{n}{m}(rj)^{[k-1,n-m]}S^{(\alpha,k)}(m,i,r). \tag{2.2}$$

In the limit when $k\to 1$, (2.2) reduces to

$$\Delta_{\alpha,r}^{i}(\alpha+rj)^n=i!\sum_{s=0}^{n}\binom{n}{s}(rj)^{n-s}S^{\alpha}(s,i,r), \tag{2.3}$$

which provides the corrected form of formula (3.2) in Shrivastava's paper [7]

For $j=0$, (2.2) yields

$$\Delta_{\alpha,r}^{i}\alpha^{[k-1,n]}=i!\,S^{(\alpha,k)}(n,i,r), \tag{2.4}$$

whence

$$\begin{aligned}\sum_{n=0}^{\infty}S^{(\alpha,k)}(n,m,r)\frac{t^n}{n!}&=\frac{1}{m!}\Delta_{\alpha,r}^{m}\sum_{n=0}^{\infty}\left(\frac{\alpha}{k-1}\right)_n(k-1)^n\frac{t^n}{n!}\\&=\frac{1}{m!}\Delta_{\alpha,r}^{m}[1-(k-1)t]^{-a/(k-1)}\\&=\sum_{j=0}^{m}\frac{(-1)^{m-j}}{m!}\binom{m}{j}[1-(k-1)t]^{-(a+rj)/(k-1)}\end{aligned}$$

(By (2.1).

Thus

$$(2.5)\quad \sum_{n=0}^{\infty} S^{(\alpha,k)}(n,m,r)\frac{t^n}{n!}=\frac{(-1)^m}{m!}[1-(k-1)t]^{-\alpha/(k-1)}[1-\{1-(k-i)t\}^{-r/(k-1)}]^m,\quad k\neq 1.$$

The generating relation (2.5) can also be derived by using (1.7).

Now using (2.5), we obtain

$$(2.6)\quad S^{(\alpha,k)}(n,m,r)=\sum_{s=0}^{n}(\alpha-\beta)^{[k-1,s]}\binom{n}{s}S^{(\beta,k)}(n-s,m,r),$$

and

$$(2.7)\quad S^{(\alpha+\beta,k)}(n,m+p,r)=\binom{m+p}{m}^{-1}\sum_{s=0}^{n}\binom{n}{s}S^{(\alpha,k)}(n-s,m,r)\,S^{(\beta,k)}(s,p,r),$$

which can be extended in the form

$$(2.8)\quad \begin{aligned}&S^{(\alpha_1+\cdots+\alpha_s,k)}(n,m_1+\cdots+m_s,r)\\ &=\frac{m_1!\cdots m_s!}{(m_1+\cdots+m_s)!}\sum_{n_1+\cdots+n_s=n}\prod_{j=1}^{s}\frac{n!}{n_j!}S^{(\alpha_j,k)}(n_j,m_j,r).\end{aligned}$$

3. Recurrence relations

Since

$$(3.1)\quad \Delta_{\alpha,r}^{i+j}=\Delta_{\alpha,r}^{i}\Delta_{\alpha,r}^{j},$$

therefore, using (2.4), we get

$$(3.2)\quad \Delta_{\alpha,r}^{i}S^{(\alpha,k)}(n,j,r)=(j+1)_i S^{(\alpha,k)}(n,i+j,r).$$

On interchanging the roles of i and j in (3.2), it is obvious that

$$(3.3)\quad j!\,\Delta_{\alpha,r}^{i}S^{(\alpha,k)}(n,j,r)=i!\,\Delta_{\alpha,r}^{j}S^{(\alpha,k)}(n,i,r).$$

Now applying (2.1) in (3.2), we obtain

$$(3.4)\quad (i+1)_j S^{(\alpha,k)}(n,i+j,r)=\sum_{s=0}^{j}(-1)^{j-s}\binom{j}{s}S^{(\alpha+rs,k)}(n,i,r),$$

which, for $j=1$, gives

$$(3.5)\quad (i+1)S^{(\alpha,k)}(n,i+1,r)+S^{(\alpha,k)}(n,i,r)=S^{(\alpha+r,k)}(n,i,r).$$

For different values of j, we shall get various recurrence relations, which can also be obtained by using (2.5) instead of (3.4).

From (2.5), we also obtain

$$(3.6)\quad S^{(\alpha-k+1,k)}(n,i,r)=S^{(\alpha,k)}(n,i,r)-n(k-1)S^{(\alpha,k)}(n-1,i,r).$$

Now from (1.7) we derive

$$S^{(\alpha+k-1,\,k)}(n,\,m,\,r)=n!\sum_{s=0}^{n}\frac{(k-1)^{n-s}}{s!}\,S^{(\alpha,\,k)}(s,\,m,\,r), \tag{3.7}$$

$$S^{(\alpha-k+1,\,k)}(n+1,\,m,\,r)=(\alpha-k+1)\,S^{(\alpha,\,k)}(n,\,m,\,r)+S^{(\alpha+r,\,k)}(n,\,m-1,\,r) \tag{3.8}$$

and

$$\begin{aligned} &S^{(\alpha,\,k)}(n+1,\,m,\,r)\\ &=[\alpha+r\,m+n\,(k-1)]\,S^{(\alpha,\,k)}(n,\,m,\,r)+r\,S^{(\alpha,\,k)}(n,\,m-1,\,r). \end{aligned} \tag{3.9}$$

The relation (3.8) can also be obtained by differentiating (2.5) with respect to it, and then equating the coefficients of t^n on both the sides.

From (3.9), we derive the following congruences (mod 2)

$$S^{(\alpha,\,k)}(n+1,\,2m,\,r)\equiv[\alpha+n\,(k-1)]\,S^{(\alpha,\,k)}(n,\,2m,\,r)+r\,S^{(\alpha,\,k)}(n,\,2m-1,\,r), \tag{3.10}$$

$$\begin{aligned} &S^{(\alpha,\,k)}(n+1,\,2m+1,\,r)\equiv\\ &\equiv[\alpha+r+n\,(k-1)]\,S^{(\alpha,\,k)}(n,\,2m+1,\,r)+r\,S^{(\alpha,\,k)}(n,\,2m,\,r), \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &S^{(\alpha,\,k)}(n+1,\,2m+1,\,r)\equiv[\alpha+r+n\,(k-1)]\,S^{(\alpha,\,k)}(n,\,2m+1,\,r)\\ &+r\,[\alpha+(n-1)\,(k-1)]\,S^{(\alpha,\,k)}(n-1,\,2m,\,r)+r^2\,S^{(\alpha,\,k)}(n-1,\,2m-1,\,r). \end{aligned} \tag{3.12}$$

4. Miscellaneous results

An appeal to (2.4) shows that

$$\begin{aligned} &e^{t\,\Delta\alpha,\,r}\{\alpha^{[k-1,\,n]}\}=\sum_{m=0}^{\infty}S^{(\alpha,\,k)}(n,\,m,\,r)\,t^m\\ &=e^{-t}\,(k-1)^n\left(\frac{\alpha}{k-1}\right)_n {}_MF_M\left[\begin{matrix}\Delta\left(M,\,\dfrac{\alpha}{k-1}+n\right);\\ \Delta\,(M,\,n);\end{matrix}\;t\right], \end{aligned} \tag{4.1}$$

provided that $M=r/(k-1)$ is a positive integer, and $\Delta\,(M,\,n)$ stands for the set of M parameters $\dfrac{n}{M},\,\dfrac{n+1}{M},\,\ldots,\,(n+1-M)/M.$

Replacing t by px^r, and using the author's result [2, (4.1)], we derive

$$e^{px^r}\,\Delta_{\alpha r}\,\{\alpha^{[k-1,\,n]}\}=x^{n(1-k)}\,T_n^{(\alpha,\,k)}(x,\,r,\,-p), \tag{4.2}$$

from which we can further write

$$\begin{aligned} e^{-px^r\,\Delta\,\alpha,\,r}\{T_n^{(\alpha,\,k)}(x,\,r,\,q)\}&=e^{-qx^r\,\Delta\,\alpha,\,r}\,\{T_n^{(\alpha,\,k)}(x,\,r,\,p)\}\\ &=T_n^{(\alpha,\,k)}(x,\,r,\,p+q). \end{aligned} \tag{4.3}$$

Now from the relation (cf. [8], p. 77; see also [3])

$$T_n^{(\alpha,k)}(x,r,p)=x^{(k-1)n}\sum_{m=0}^{n}\frac{p^m x^{rm}}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(\alpha+rj)^{[k-1,n]},$$

we derive the following relation for generalized Stirling polynomials

$$T_n^{(\alpha,k)}(x,r,-p)=x^{(k-1)n}e^{-px^r}\sum_{j=0}^{\infty}\frac{p^j x^{rj}}{j!}(\alpha+rj)^{[k-1,n]}. \tag{4.4}$$

It is fairly well known that if

$$g(r)=\sum_{j=0}^{r}\binom{r}{j}f(j) \qquad (r=0,1,2,\ldots), \tag{4.5}$$

then

$$f(r)=\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}g(j). \tag{4.6}$$

Now applying (4.5) and (4.6) in (1.7), we obtain

$$(\alpha+rj)^{[k-1,n]}=\sum_{m=0}^{j}\binom{j}{m}m!\,S^{(\alpha,k)}(n,m,r), \tag{4.7}$$

which can be used here in (4.4) to get (1.8)

Further, operating on both the sides by $\Delta_{\alpha,r}^{i}$ and making an appeal to (3.2), the result (4.7) gives

$$\Delta_{\alpha,r}^{i}(\alpha+rj)^{[k-1,n]}=\sum_{m=0}^{j}\binom{j}{m}(m+i)!\,S^{(\alpha,k)}(n,m+i,r), \tag{4.8}$$

which is different from (2.2).

For $j=0$ and 1, the above result gives (2.4) and (3.5), respectively, but for $i=1$ it gives

$$(\alpha+r(j+1))^{[k-1,n]}-(\alpha+rj)^{[k-1,n]}=\sum_{m=0}^{j}\binom{j}{m}(m+1)!\,S^{(\alpha,k)}(n,m+1,r), \tag{4.9}$$

which for $r=0$ yields

$$\sum_{m=0}^{j}\binom{j}{m}(m+1)!\,S^{(\alpha,k)}(n,m+1,0)=0. \tag{4.10}$$

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