PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 22 (36), 1977 pp. 271—281

ON THE CONVERGENCE OF CERTAIN SEQUENCES AND SOME APPLICATIONS, II

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(Communicated October 15, 1975)

Abstract. In this paper we prove a somewhat more general theorem on the convergence of sequences, and prove a fixed point theorem for operators on ordered sets.

1. Introduction

1.0. In [3] we have proved the following lacalization theorem

Theorem *T*. *Let* (X, ρ) *be a complete metric space and let* T *be a mapping of* X^k *to* X *satisfying the condition*

$$\text{(A)} \qquad \rho[T(u_1, \ldots, u_k),\ T(u_2, \ldots, u_{k+1})] \leqq f(\alpha_1 \rho[u_1, u_2], \ldots, \alpha_k \rho[u_k, u_{k+1}])$$

for every $u_1, u_2, \ldots, u_{k+1} \in X$, *where* $f(\alpha_1, \ldots, \alpha_k) \in [0, 1)$, *and the mapping* $f: R_+^k \to R_+$ *(k given natural number) has the property (F). Then:*

(a) *There exist a fixed point* $\xi \in X$ *of the mapping* $\mathcal{T}(x) \overset{\text{def}}{=\!=} T(x, \ldots, x)$, *and it is unique when* $f(\alpha_1, 0, \ldots, 0) + \cdots + f(0, \ldots, 0, \alpha_k) < 1$;

(b) ξ *is the limit of the sequence* (x_n) *satisfying*

$$\text{(1)} \qquad x_{n+k} = T(x_n, \ldots, x_{n+k-1}), \qquad n \in N,$$

independently of initial values.

(c) *The rapidity of convergence of the sequence* (x_n) *to the point* ξ *is evaluated by*

$$\rho[x_{n+k}, \xi] \leqq \theta^n (1-\theta)^{-1} \max_{i=1, \ldots, k} (\rho[x_i, x_{i+1}]\, \theta^{-i})\}; \ \theta \in (0, 1), \qquad n \in N.$$

We say that the mapping $f: R_+^k \to R_+$ $(k \in N)$ has the property (F) iff is increasing, semihomogeneous (that is to say, for all $\delta \geqq 0$ one has $f(\delta x_1, \ldots, \delta x_k) \leqq \delta f(x_1, \ldots, x_k))$ and $g(x) = f(\alpha_1 x, \ldots, \alpha_k x^k)$ be continuous at the point $x = 1$, where α_i $(i = 1, \ldots, k)$ are nonegative real constants.

In the second paragrafh of this paper we shall extend this result to the ordered sets, but then this localization's Theorem has a different property. It is called Theorem 1.

In the paper [10] was considered this problem. But the establishment of this problem was not good, and therefore was are turning again on this in Theorem 1.

1.1. In other words, we shall examine the convergence of sequences in complete metric spaces which satisfy certain recurrent relations. Namely, let X be a complete metric space and let $T_i: X^{pk} \to X$ ($i=1, \ldots, k$; p is a natural number). We give sufficient conditions for the sequences $(x_n^{(1)}), \ldots, (x_n^{(k)})$ which satisfy the following relations

$$x_{n+p}^{(i)} = T_i\left(x_n^{(1)}, \ldots, x_{n+p-1}^{(1)}, \ldots, x_n^{(k)}, \ldots, x_{n+p-1}^{(k)}\right), \; x_i^{(j)} \in X,$$

$i=1, \ldots, k$, to converge in X.

In order to simplify the techique we shall limit ourselves to the case $k=2$, $p=2$. The general result can be easily seen from this special case.

This is, therefore, an extension of the results obtained in [1], [3], [4], [6], [7] and [8].

2. A localization theorem on ordered sets

2.1. Let $\mathcal{O}$ (see [8] or [9]) through this section be a set having the following properties:

1) $\mathcal{O}$ is a partially ordered set by the relation $\leqslant$ and there exists $\theta \in \mathcal{O}$ such that $\theta \leqslant u$ for every $u \in \mathcal{O}$.

2) For every $u, v \in \mathcal{O}$ is defined $u+v \in \mathcal{O}$, such that:

(a) $u+v=v+u$, $u+\theta=u$ ($u, v \in \mathcal{O}$),

(b) if $u, v, w \in \mathcal{O}$ and $u \leqslant v$, then $u+w \leqslant v+w$,

(c) if $u+v \leqslant w$ then $u \leqslant w$ ($u, v, w \in \mathcal{O}$).

3) For every nonincreasing sequence (u_n) there exists the unique element u of $\mathcal{O}$ called the limit of (u_n) all signed by $u = \lim_{n\to\infty} u_n$ (alternative designation $u_n \downarrow u$), such that:

(a) if $u_n=u$ for every $n \in N$ then $u_n \downarrow u$;

(b) if $u_n=u$ and $v_n \downarrow v$ then $u_n+v_n \downarrow u+v$;

(c) if $u_n \downarrow u$, $v_n \downarrow v$ and $u_n \leqslant v_n$ then $u \leqslant v$;

(d) the limit of (u_n) is invariant with respect to the initial conditions.

We remark that 3) is especially realized if in $\mathcal{O}$ is introduced the usual ordered topological structure and each subset of $\mathcal{O}$ from the upper side bounded has its *supremum*, the term of limit having its standard meaning.

Let $f: \mathcal{O}^k \to \mathcal{O}$ ($k \in N$). A function f is nondecreasing if from $u_i \leqslant v_i$ ($u_i, v_i \in \mathcal{O}$; $i=1, 2, \ldots, k$) follows $f(u_1, \ldots, u_k) \leqslant f(v_1, \ldots, v_k)$. Let the sequences $u_n^i \downarrow u^i$ ($u_n^i \in \mathcal{O}$; $n \in N$, $i=1, 2, \ldots, k$). If $f(u_n^1, \ldots, u_n^k) \downarrow f(u^1, \ldots, u^k)$, we say that f is continuous with respect to sequences.

The pair (E, ρ), where E is a set and $\rho: E^2 \to \mathcal{O}$, will be called a metric space if the standard properties of the metric function are fulfilled. If every sequence (x_n), $x_n \in E$, which satisfies the condition $\rho[x_n, x_m] \leqslant a_n$ $(m, n \in N)$, where $a_n \downarrow \theta$ $(a_n \in \mathcal{O})$, converges in E then E is a complete metric space.

Let $u, v \in \mathcal{O}$ and $u \leqslant v$. The segment $[u, v] \subset \mathcal{O}$ is the set of all elements $w \in \mathcal{O}$, such that $u \leqslant w \leqslant v$. Let $S(z, r) \overset{\text{def}}{=} \{u \in E : \rho[u, z] \leqslant r\}$.

2.2. Now we can prove our main result.

Theorem 1. *Let E be a complete metric space and let the sphere $S(z, r) \subset E$. Furthermore, let $T: E^k \to E$ $(k \in N)$ such that*

(B) $$\rho[T(u_1, \ldots, u_k), T(v_1, \ldots, v_k)] \leqslant f(\rho[u_1, v_1], \ldots, \rho[u_k, v_k])$$

holds for every $u_i, v_i \in S(z, r)$, $i = 1, \ldots, k$; and $f: [\theta, h]^k \to \mathcal{O}$ $([\theta, h] \subset \mathcal{O})$ is non-decreasing and continuous with respect to the sequences such that the equation

(2) $$x = f(x, \ldots, x), \quad (x \in [\theta, h]),$$

has the unique solution θ. Also, let there exist $q \in [\theta, h]$ such that

$$\rho[z, T(z, \ldots, z)] \leqslant q, \quad q + f(r, \ldots, r) \leqslant r,$$

where $f(y, \ldots, y) \leqslant y$ for $2r \leqslant y$.

Then:

(a) $(\exists \xi \in S(z, r))\, T(\xi, \ldots, \xi) = \xi$,

(b) *ξ is the limit of the sequence (x_n) defined by* (1) *where $x_1, \ldots, x_k \in S(z, r)$ are arbitrarly chosen,*

(c) *The unique solution of the equation $x = T(x, \ldots, x)$ is $\xi = \lim_{n \to \infty} x_n$, and*

(d) $\rho[x_{n+k}, \xi] \leqslant A_{n+k}(y)$,

where

$$A_i(y) \overset{\text{def}}{=} f_i(y, \ldots, y) = f(f_{i-k}(y, \ldots, y), \ldots, f_{i-1}(y, \ldots, y)), \qquad (i > k),$$
$$A_i(y) \overset{\text{def}}{=} f_i(y, \ldots, y) = y, \quad (i \leq k).$$

Proof. Let $u_1, \ldots, u_k \in S(z, r)$ then

$$\begin{aligned} \rho[T(u_1, \ldots, u_k), z] &\leqslant \rho[T(u_1, \ldots, u_k), T(z, \ldots, z)] + \rho[T(z, \ldots, z), z] \\ &\leqslant f(\rho[u_1, z], \ldots, \rho[u_k, z]) + q \\ &\leqslant f(r, \ldots, r) + q \leqslant r, \end{aligned}$$

we conclude that the function T maps $S(z, r)^k$ into $S(z, r)$.

Let $x_n \in S(z, r)$, $n \in N$; we have

$$\rho[x_i, x_{m+i}] \leqslant \rho[x_i, z] + \rho[z, x_{m+i}] \leqslant 2r \leqslant y \quad (i = 1, \ldots, k;\ m \in N).$$

In other words, let us suppose that

(3) $$\rho[x_{n+i}, x_{n+m+i}] \leqslant f_{n+i}(y, \ldots, y) \overset{\text{def}}{=} A_{n+i}(y), \qquad (i = 1, \ldots, k),$$

holds, where

$$(4)\quad \begin{aligned} &f_i(y,\dots,y)=f(f_{i-k}(y,\dots,y),\dots,f_{i-1}(y,\dots,y))\stackrel{\text{def}}{=}A_i(y), \qquad (i>k),\\ &f_i(y,\dots,y)=y\stackrel{\text{def}}{=}A_i(y), \qquad (i\le k).\end{aligned}$$

Then

$$\begin{aligned}\rho[x_{n+k+1},x_{n+k+m+1}]&=\rho[T(x_{n+1},\dots,x_{n+k}),T(x_{n+m+1},\dots,x_{n+k+m})]\\ &\leqslant f(\rho[x_{n+1},x_{n+m+1}],\dots,\rho[x_{n+k},x_{n+m+k}])\\ &\leqslant f(f_{n+1}(y,\dots,y),\dots,f_{n+k}(y,\dots,y))\\ &\stackrel{\text{def}}{=}f_{n+k+1}(y,\dots,y)\stackrel{\text{def}}{=}A_{n+k+1}(y),\end{aligned}$$

and (3) holds for every $n\in N$.

Also, we have $A_1(y)=\cdots=A_k(y)=y$.

If we suppose $A_{n+i}(y)\leqslant A_{n+i-1}(y)$, $(i=1,\dots,k)$, then

$$A_{n+k+1}(y)=f(A_{n+1}(y),\dots,A_{n+k}(y))\leqslant f(A_n(y),\dots,A_{n+k-1}(y))=A_{n+k}(y).$$

Therefore, the sequence $(A_n(y))$ is nonincreasing and $\theta\leqslant A_n(y)$, i.e. $(A_n(y))$ is convergent.

Let $\alpha=\lim\limits_{n\to\infty}A_n(y)$, and let $n\to\infty$ in (4). Then α satisfies the equation $x=f(x,\dots,x)$ and from (2) follows $\alpha=\theta$, i.e. $\lim\limits_{n\to\infty}A_n(y)=\theta$. Hence, the generalized metric space E being complete, there exist $\xi=\lim\limits_{n\to\infty}x_n\in S(z,r)$.

Making $m\to\infty$ in (3), we get

$$\rho[x_{n+k},\xi]\leqslant A_{n+k}(y).$$

On the other hand, we have

$$\begin{aligned}\rho[\xi,T(\xi,\dots,\xi)]&\leqslant\rho[\xi,x_{n+k}]+\rho[T(x_n,\dots,x_{n+k-1}),T(\xi,\dots,\xi)]\\ &\leqslant\rho[\xi,x_{n+k}]+f(\rho[x_n,\xi],\dots,\rho[x_{n+k-1},\xi])\\ &\leqslant A_{n+k}(y)+f(A_n(y),\dots,A_{n+k-1}(y))\\ &\leqslant 2A_{n+k}(y).\end{aligned}$$

Hence, for $n\to\infty$, we get that $\xi=\lim x_n$ satisfies the equation $x=T(x,\dots,x)$.

If ξ^* is an element of $S(z,r)$ such that $\xi^*=T(\xi^*,\dots,\xi^*)$ then

$$\rho[\xi^*,x_i]\leqslant\rho[\xi^*,z]+\rho[z,x_i]\leqslant 2r\leqslant y=A_i(y), \qquad (i=1,\dots,k)$$

If

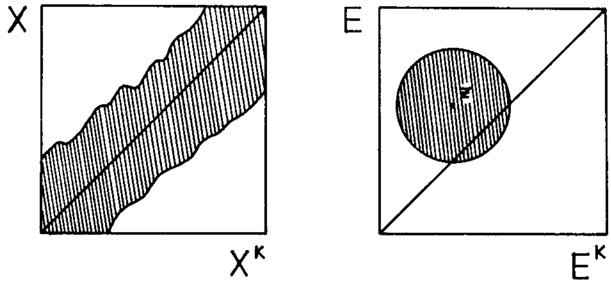
$$(5)\qquad \rho[\xi^*,x_{n+i}]\leqslant A_{n+i}(y) \qquad (i=1,2,\dots,k)$$

then

$$\begin{aligned}\rho[\xi^*,x_{n+k+1}]&=\rho[T(\xi^*,\dots,\xi^*),T(x_{n+1},\dots,x_{n+k-1})]\\ &\leqslant f(\rho[\xi^*,x_{n+1}],\dots,\rho[\xi^*,x_{n+k}])\\ &\leqslant f(A_{n+1}(y),\dots,A_{n+k}(y))=A_{n+k+1}(y)\end{aligned}$$

So (5) holds for every $n \in N$. Making $n \to \infty$ in (5) we have $\rho[\xi^*, \xi] = \theta$ which is impossible. The proof is complete.

Remark. 1) In the next picture we give the geometric intrepretation of the localization which is realized by the conditions (*A*) and (*B*) by the previous theorems.



(A) (B)

2) Under the conditions of the Theorem 1 substituting the condition (*B*) with:

$$\rho[T_n(u_1, \ldots, u_k), T_{n+1}(v_1, \ldots, v_k)] \leqslant f(\rho[u_1, v_1], \ldots, \rho[u_k, v_k]),$$

where $T_n: E^k \to E$, and condition $\rho[z, T(z, \ldots, z)] \leqslant q$ with $\rho[z, T_n(z, \ldots, z)] \leqslant q$; affirmation of Theorem 1 is valuable, but relates to the sequence

$$x_{n+k} = T_n(x_n, \ldots, x_{n+k-1}), \qquad n \in N.$$

3. On the convergence of certain sequences

3.1. Let $(\mathcal{O}, \leqslant)$ be a set ordered by the ordered relation $\leqslant$, and let $\circ$ be a binary operation on $\mathcal{O}$ with inverse operation $\alpha(\circ)$ and satisfying the conditions

(a) $(\forall, y, z \in \mathcal{O})\, x \leqslant z \circ y \Leftrightarrow x\, \alpha(\circ)\, y \leqslant z$,

(b) $(\forall, y, z \in \mathcal{O})\, x \leqslant y \Rightarrow z \circ x \leqslant z \circ y$.

In [5] we have proved the following

Proposition *T*. Let $(\mathcal{O}, \leqslant)$ be a set ordered by the ordered relation $\leqslant$ and having the property: For any two elements a and b of $\mathcal{O}$, the set$\{a, b\}$ has an upper bound. and let $f_n: \mathcal{O}^{2n-2} \to \mathcal{O}\, (n \in N)$ be a monotonically increasing and semihomogeneous mapping, and (x_n), (y_n), (X_n), (Y_n) sequences satisfying

$$x_n, y_n \leqslant f_n(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1})$$

$$f_n(X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1}) \leqslant X_n, Y_n.$$

Then there exist elements $\mathscr{L}_1, \mathscr{L}_2 \in O$ such that

$$x_n \leqslant \mathscr{L}_1 \circ X_n, \; y_n \leqslant \mathscr{L}_2 \circ Y_n \qquad (n=1, 2, \ldots).$$

3.2. Theorem. 2. *Let (X, ρ) be a complete metric space and let $T_n, U_n : X^4 \to X$ be sequences of functions such that*

$$\max\{\rho[T_n(u_1, u_2, u_3, u_4), T_{n-1}(u_3, u_4, u_5, u_6)], \rho[U_n(u_1, u_2, u_3, u_4), U_{n-1}(u_3, u_4, u_5, u_6)]\}$$
(C)
$$\leqq \alpha \max\{\rho[u_1, u_3], \rho[u_2, u_4] \rho[u_3, u_5], \rho[u_4, u_6]\} + a_n,$$

for every $u_1, u_2, \ldots, u_6 \in X$, where $\alpha \in [0, 1)$ and the series $\sum_{\nu=1}^{\infty} a_\nu \, (a_\nu > 0)$ converge.

Then:

(a) *Sequences of functions $T_n(u, v, u, v)$, $U_n(u, v, u, v)$ converge uniformly to functions $T(u, v, u, v)$, $U(u, v, u, v)$;*

(b) *Sequences (x_n), (y_n) given by*

$$x_{n+2} = T_n(x_n, y_n, x_{n+1}, y_{n+1}), \; y_{n+2} = U_n(x_n, y_n, x_{n+1}, y_{n+1}), \quad (n \in N), \tag{6}$$

where the elements x_1, x_2, y_1, y_2 are arbitrarily chosen, converge in X;

(c) *Solution of the system*

$$x = T(x, y, x, y), \; y = U(x, y, x, y)$$

is $x = \lim x_n$, $y = \lim y_n$.

Proof. (a). Putting $u_1 = u_3 = u_5 = u$, $u_2 = u_4 = u_6 = v$ in (C) we get

$$\max\{\rho[T_n(u, v, u, v), T_{n-1}(u, v, u, v)], \rho[U_n(u, v, u, v), U_{n-1}(u, v, u, v)]\} \leqq a_n,$$

which proves the uniform convergence of sequences $T_n(u, v, u, v)$, $U_n(u, v, u, v)$.

Proof. (b), (c). Denote $\rho[x_n, x_{n+1}]$ by Δ_n, and $\rho[y_n, y_{n+1}]$ by ∇_n. Then from (C),

$$\max\{\Delta_{n+2}, \nabla_{n+2}\} \leqq \alpha \max\{\Delta_n, \nabla_n, \Delta_{n+1}, \nabla_{n+1}\} + a_n. \tag{7}$$

Denote $\limsup \Delta_n$ by Δ, and $\limsup \nabla_n$ by ∇.

Then from (7) we get

$$\max\{\Delta, \nabla\} \leqq \alpha \max\{\Delta, \nabla, \Delta, \nabla\} = \alpha \max\{\Delta, \nabla\},$$

and therefore, $\lim \max\{\Delta_n, \nabla_n\} = 0$ which together implies

$$\max\{\rho[x_n, x_{n+p}], \rho[y_n, y_{n+p}]\} \to 0, \qquad n \to \infty,$$

i.e. we see that the sequences (x_n), (y_n) are convergent. Let $x = \lim x_n$, $y = \lim y_n$.

Then

$$\rho[x_{n+2}, T(x, y, x, y)] + \rho[y_{n+2}, U(x, y, x, y)] \leq$$

$$\leq \rho[T_n(x_n, y_n, x_{n+1}, y_{n+1}), T_{n-1}(x_{n+1}, y_{n+1}, x, y)] + \rho[T_{n-1}(x_{n+1}, y_{n+1}, x, y), T(x, y, x, y)] + \rho[T_{n-2}(x, y, x, y), T(x, y, x, y)] + \rho[U_n(x_n, y_n, x_{n+1}, y_{n+1}), U_{n-1}(x_{n+1}, y_{n+1}, x, y)] + \rho[U_{n-1}(x_{n+1}, y_{n+1}, x, y), U_{n-2}(x, y, x, y)] + \rho[U_{n-2}(x, y, x, y), U(x, y, x, y)] \leq 2\max\{\rho[T_n(x_n, y_n, x_{n+1}, y_{n+1}), T_{n-1}(x_{n+1}, y_{n+1}, x, y)], \rho[U_n(x_n, y_n, x_{n+1}, y_{n+1}), U_{n-1}(x_{n+1}, y_{n+1}, x, y)]\} + 2\max\{\rho[T_{n-1}(x_{n+1}, y_{n+1}, x, y), T_{n-2}(x, y, x, y)], \rho[U_{n-1}(x_{n+1}, y_{n+1}, x, y), U_{n-2}(x, y, x, y)]\} + \rho[T_{n-2}(x, y, x, y), T(x, y, x, y)] + \rho[U_{n-2}(x, y, x, y), U(x, y, x, y)] \leq 2\alpha\max\{\rho[x_n, x_{n+1}], \rho[y_n, y_{n+1}], \rho[x_{n+1}, x], \rho[y_{n+1}, y]\} + 2\alpha\max\{\rho[x_{n+1}, x], \rho[y_{n+1}, y], 0, 0\} + \rho[T_{n-2}(x, y, x, y), T(x, y, x, y)] + \rho[U_{n-2}(x, y, x, y), U(x, y, x, y)] + 2a_n.$$

i.e.

$$\lim \rho[x_{n+2}, T(x, y, x, y)] = 0, \lim \rho[y_{n+2}, U(x, y, x, y)] = 0$$

or

$$\lim x_n = x = T(x, y, x, y),$$
$$\lim y_n = y = U(x, y, x, y).$$

The proof of the Theorem is complete.

3.3. We obtain now the following theorem for the convergence of sequences, the proof of which is essentially the same to that of Theorem 1. and proposition T:

Theorem 3. *Let (X, ρ) be a complete metric space and let $T_n, U_n: X^4 \to X$ be sequences of functions such that*

$$\max\{\rho[T_n(u_1, u_2, u_3, u_4), T_{n-1}(u_3, u_4, u_5, u_6)], \rho[U_n(u_1, u_2, u_3, u_4), U_{n-1}(u_3, u_4, u_5, u_6)]\}$$
$$\leq f(\alpha_1\rho[u_1, u_3], \alpha_2\rho[u_2, u_4], \alpha_3\rho[u_3, u_5], \alpha_4\rho[u_4, u_6]) + a_n,$$

for every $u_1, u_2, \dots, u_6 \in X$ and $\sum_{\nu=1}^{\infty} a_n < \infty$ $(a_n > 0)$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are nonnegative fixed numbers such that $|f(\alpha_1, \dots, \alpha_4)| \in [0, 1)$ and the mapping $f: R^4 \to R$ is increasing, continuous and semihomogeneous. Then:

(a) *Sequences of functions $T_n(u, v, u, v)$, $U_n(u, v, u, v)$ converge uniformly to functions $T(u, v, u, v)$, $U(u, v, u, v)$.*

(b) *Sequences (x_n), (y_n) given by (6) have the same properties as in Theorem 2.*

4. Corollaries and examples

The Theorem 1 contains, as particular cases, a number of known results.

4.1. Corollary 1. If $(O, \preccurlyeq)$ is the set of nonnegative real numbers and

$$f(t, \ldots, t_k) = \alpha \max\{t_1, \ldots, t_k\} \qquad (t_1, \ldots, t_k \geqq 0)$$

where $\alpha \in [0, 1)$, we get the result 1 of [5].

Corollary 2. In the case $k=1$ we get Kurpel's theorem 2.2. (see [9]).

4.2. To a matrix $(C_{ik}^{(0)})$, $i, k = 1, \ldots, n$, we associate the sequence of $n-1$ matrices $(C_{ik}^{(l)})$ defined as follows

$$C_{ik}^{(l)} = \begin{cases} C_{11}^{(l-1)} C_{i+1,k+1}^{(l-1)} + C_{i+1,1}^{(l-1} C_{i,k+1}^{(l-1)}, & \text{for } i \neq k \\ C_{11}^{(l-1)} C_{i+1,k+1}^{(l-1)} - C_{i+1,1}^{(l-1)} C_{i,k+1}^{(l-1)}, & \text{for } i = k, \end{cases}$$

$l = 1, \ldots, n-1$; $i, k = 1, \ldots, n-l$.

Lemma 1. (J. Matkowsky, [1], p. 323). Let $C_{ik}^{(0)} > 0$; $i, k = 1, \ldots, n$. The system of inequalities

$$\sum_{\substack{k=1 \\ (k \neq i)}}^{n} C_{ik}^{(0)} r_k < C_{ii}^{(0)} r_i \qquad (i = 1, \ldots, n),$$

has a solution $r_i > 0$ $(i = 1, \ldots, n)$, if and only if the following inequalities hold:

$$C_{ii}^{(l)} > 0, \quad (l = 1, \ldots, n-1; \ i = 1, \ldots, n-l). \tag{8}$$

Using this Lemma J. Matkowsky was able to prove the following:

Corollary 3. (J. Matkowsky, [1], p. 323). Let (X_i, ρ_i), $i = 1, \ldots, n$, be complete metric spaces. Suppose that the transformations $T_i: X_1 \times \cdots \times X_n \to X_i$ $(i = 1, \ldots, n)$ fulfil the following conditions

$$\rho_i [T_i(u_1, \ldots, u_n), \ T_i(v_1, \ldots, v_n) \leqq \sum_{k=1}^{n} a_{ik} \rho_k [u_k, v_k]$$

where $u_k, v_k \in X_k$, $a_{ik} > 0$; i $k = 1, \ldots, n$. If the numbers $C_{ik}^{(0)}$ defined by

$$C_{ik}^{(0)} = \begin{cases} a_{ik}, & \text{for } i \neq k \\ 1 - a_{ik}, & \text{for } i = k \ (i, k = 1, \ldots, n) \end{cases} \tag{9}$$

fulfil the inequalities (8), then the system of equation

$$x_i = T_i(x_1, \ldots, x_n); \ i = 1, \ldots, n,$$

has exactly one solution $x_i \in X_i$ $(i = 1, \ldots, n)$.

Moreover $x_i = \lim_{m\to\infty} x_i^m$ $(i=1,\dots,n)$, where $x_i^0 \in X_i$ $(i=1,\dots,n)$, are arbitrarily chosen and

$$x_i^{m+1} = T_i(x_1^m,\dots,x_n^m),\ (i=1,\dots,n;\ m=0,1,\dots)$$

Lemma 2. (J. Matkowsky, [2], p. 12). Let (α_{ik}), $(i,k=1,\dots,n)$ be a non-negative matrix with characteristic roots $\lambda_1,\dots,\lambda_n$ and let $C_{ik}^{(0)}$ be defined by (9). Then conditions (8) are equivalent to the following:

$$s = \max(|\lambda_i| : i=1,\dots,n) < 1.$$

Corollary 4. Let O be the set of non-negative k-vectors with the natural ordering and let (E, ρ) be a complete metric space. Furthermore, let the metric in E^k by defined

$$D(X,Y) = \begin{bmatrix} d(x_1,y_1) \\ \vdots \\ d(x_k,y_k) \end{bmatrix},\ (x_i, y_i \in E),$$

where $X=(x_1,\dots,x_k)$, $Y=(y_1,\dots,y_k)$ and let $F:(E^k)^p \to E^k$ $(k,p \in N)$ be defined by

$$F(Y_1,\dots,Y_p) \overset{\text{def}}{=} (f_1(Y_1,\dots,Y_p),\dots,f_k(Y_1,\dots,Y_k))$$

where $f_i: E^{pk} \to E$, $Y_i \in E^k$ $(i=1,2,\dots,p)$.

Let the sequence (X_n) be defined by

$$X_{n+p} = F(X_n,\dots,X_{n+p-1}) \qquad (X_n \in E^k)$$

and let

$$f(U_1,\dots,U_p) = A_1U_1 + \dots + A_pU_p,$$

where A_i are $k\times k$ matrices with non-negative fixed elements and U_i are k-vectors $(i=1,\dots,p)$ satisfy the condition

(10) $$N(A_1+\dots+A_p) < 1$$

where N is a matrix norm. The condition (10) was replaced by: $|\lambda_j| < 1$ for each root of the equation

$$\det(\lambda^p I - \lambda^{p-1}A_p - \dots - A_p) = 0.$$

It is easy to see that (10) implies conditions of Theorem 1 and we conclude that the result of paper [1] is a consequence of the theorem 1.

4.3. In the special case when we are dealing with sequences of real numbers, we immediately obtain the following

Theorem 4. *Let $T, U: R^3 \to R$ (R is the set of real numbers) and let*

(D) $$\max\{|T(u_1,u_2,x)-T(u_2,u_3,y)|, |U(u_1,u_2,x)-U(u_2,u_3,y)|\} \leqq$$
$$\leqq \alpha \max\{|u_1-u_2|, |u_2-u_3|\} + |x-y|,$$

where $\alpha \in [0, 1)$. *Let the sequences* (x_n), (y_n) *be defined by*

$$x_{n+1} = T(x_n, y_n, a_n)$$
$$y_{n+1} = U(x_n, y_n, b_n), \qquad n = 1, 2, \ldots$$

(*elements* x_1, y_1 *are arbitrarily chosen*) *where:*

1) T, U *satisfy* (D),

2) *Sequences* (a_n), (b_n) *are monotonic and* $\lim a_n = a$, $\lim b_n = b$, *where* a, b *finite numbers.*

Then the sequences (x_n), (y_n) *converge and the system*

$$x = T(x, y, a), \; y = U(x, y, b)$$

has the solution $x = \lim x_n$, $y = \lim y_n$.

Proof. Put

$$T(x_n, y_n, a_n) = T_n(x_n, y_n)$$
$$U(x_n, y_n, b_n) = U_n(x_n, y_n).$$

Then from (D)

$$\max\{|T_n(x_n, y) - T_{n-1}(x_{n-1}, y_{n-1})|, \; |U_n(x_n, y_n) - U_{n-1}(x_{n-1}, y_{n-1})|\} \leqq$$
$$\leqq \alpha \max\{|x_n - x_{n-1}|, \; |y_n - y_{n-1}|\} + \max\{|a_n - a_{n-1}|, \; |b_n - b_{n-1}|\}.$$

Since $\sum \max\{|a_n - a_{n-1}|, \; |b_n - b_{n-1}|\} < \infty$ we see that the condition of the Theorem 2 is fulfilled and that, therefore, sequences (x_n), (y_n) converge. Furthermore,

$$\max\{|x_{n+1} - T(x, y, a)|, \; |y_{n+1} - U(x, y, a)|\} \to 0 \qquad (n \to \infty).$$

Corollary 5. (J. D. Kečkić, [6], p. 76). Let $T_n, U_n : X^4 \to X$ (X is complete metric space) and let

$$\text{(E)} \qquad \begin{cases} \rho[T_n(u_1, u_2, u_3, u_4), \; T_{n-1}(u_3, u_4, u_5, u_6)] \leqq \\ \leqq a_1 \rho[u_1, u_3] + a_2 \rho[u_2, u_4] + a_3 \rho[u_3, u_5] + a_4 \rho[u_4, u_6] + \alpha_n \\ \rho[U_n(u_1, u_2, u_3, u_4), \; U_{n-1}(u_3, u_4, u_5, u_6)] \leqq \\ \leqq b_1 \rho[u_1, u_3] + b_2 \rho[u_2, u_4] + b_3 \rho[u_3, u_5] + b_4 \rho[u_4, u_6] + \beta_n, \end{cases}$$

for every $u_1, u_2, \ldots, u_6 \in x$, where the non-negative numbers a_i, b_i $(i = 1, 2, 3, 4)$ satisfy the condition

$$\text{(F)} \qquad \max\{a_1 + b_1 + a_3 + b_3, \; a_2 + b_2 + a_4 + b_4\} < 1,$$

whereas the series $\sum \alpha_n$, $\sum \beta_n$ converge. Then:

1) Sequences of functions $T_n(u, v, u, v)$, $U_n(u, v, u, v)$ converge uniformly to functions $T(u, v, u, v)$, $U(u, v, u, v)$.

2) Sequences (x_n), (y_n) given by (1), converge to $x, y \in X$, where (x, y) is the only solution of the system.

$$x = T(x, y, x, y), \quad y = U(x, y, x, y).$$

Comparing Theorem 2 with the above theorem proved in [6] one might seek the relationship between condition (C) and (E). Since condition (E) implies the condition of (C) our Theorem 2 is the generalization of Theorem 1 of [6]. The following example shows that a (C) need not be an (E).

Example. Let $X = R$ and

$$x_{n+2} = 1/3\, x_n + 1/12\, y_n + 1/4\, x_{n+1} + 1/10\, y_{n+1} + \frac{4}{15} - \frac{1}{2n}, \tag{11}$$

$$y_{n+2} = 1/3\, x_n + 1/4\, y_n + 1/12\, x_{n+1} + 1/10\, y_{n+1} + \frac{1}{5} + \frac{1}{2n}.$$

In this case (F) is not satisfied since

$$\max\{1/3 + 1/3 + 1/4 + 1/12,\ 1/12 + 1/4 + 1/10 + 1/10\} = 2/3 + 1/3 = 1.$$

However, (C) is satisfied since

$$\max\{1/3 + 1/12 + 1/4 + 1/10,\ 1/3 + 1/4 + 1/12 + 1/10\} = 23/30 \in [0, 1),$$

and therefore the sequences (x_n), (y_n) defined by (11) converge to $x = 27/25$, $y = 1$.

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