PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 22 (36), 1977, pp. 267—269

A GENERALIZED DIRECT PRODUCT OF GRAPHS

*Risto Šokarovski**

(Received February 28. 1977)

We consider only finite, undirected graphs without loops or multiple edges.

There are many n-ary ($n \geqslant 2$) operations on graphs, the result of which is a graph whose vertex set is equal to the Cartesian product of vertex sets of graphs over which the operation is made. In this paper we shall define a generalized direct product of graphs containing, as special cases, all the operations of the mentioned type considered in the literature up to now. Some properties of the operation introduced are studied.

Let $n \geqslant 2$ be an integer. Let B be a set of n-tuples $(\beta_1, \ldots, \beta_n)$ of symbols 1, 0, -1, which does not contain n-tuple $(0, \ldots, 0)$. A set B is called the n-ary basis.

Definition. The generalized direct product (with the basis B) of graphs $G_1, \ldots, G_n$ (denoted by $GDP(B; G_1, \ldots, G_n)$) is a graph, whose vertex set is the Cartesian product of the vertex sets of graphs $G_1, \ldots, G_n$ and in which two vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent if and only if there is an n-tuple $(\beta_1, \ldots, \beta_n)$ in B, such that

(i) x_i is adjacent to y_i in G_i when $\beta_i = 1$;

(ii) $x_i = y_i$ when $\beta_i = 0$;

(iii) $x_i \neq y_i$ and x_i is not adjacent to y_i when $\beta_i = -1$.

If B is empty, then $GDP(B; G_1, \ldots, G_n)$ has no edges.

If B contains only n-tuples with the symbols 1 and 0, $GDP(B; G_1, \ldots, G_n)$ is reduced to the $NEPS$ (non-complete extended p-sum of graphs, [2]) with the same basis of the same graphs. As known, the $NEPS$ contains, as special cases, the product, the sum, the strong product and the p-sum of graphs. In [4] the $NEPS$ was rediscovered under the name C-product.

For any Boolean function $f: \{0, 1\}^n \to \{0, 1\}$, a Boolean function $f(G_1, \ldots, G_n)$ of graphs $G_1, \ldots, G_n$ is defined in [1], [2]. Let $F = \{(e_1, \ldots, e_n) \mid f(e_1, \ldots, e_n) = 1\}$ and $B = \{(2e_1 - 1, \ldots, 2e_n - 1) \mid (e_1, \ldots, e_n) \in F\}$. Then $f(G_1, \ldots, G_n) = GDP(B; G_1, \ldots, G_n)$.

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* Risto Šokarovski, Master of Science, Assistant of the Mathematical Institute, Skopje, died suddenly on December 6, 1976 at the age of 32. This article has been translated from the Macedonian original and modified by D.M. Cvetković.

For $n=2$ there are 256 types of the *GDP* and they coincide with 256 graph products considered in [3]. So, the lexicographical product of graphs is also covered by the *GDP*.

For any $n \geqslant 2$ there exist exactly 2^{3^n-1} generalized direct graph products with n-ary bases.

The *GPD* with an n-ary basis B is a commutative n-ary operation if and only if B contains, together with any n-tuple $(\beta_1, \ldots, \beta_n)$, all the permutations of $(\beta_1, \ldots, \beta_n)$, too.

All these statements can be verified in a simple way using the *GDP* definition.

Let $d(x)$ denote the degree of vertex x and let $\beta=(\beta_1, \ldots, \beta_n)$; p_i denotes the number of vertices of the graph G_i.

Lemma 1. *Vertex $(x_1, \ldots, x_n)$ in $GDP(B;\, G_1, \ldots, G_n)$ has the degree defined by*

$$d(x_1, \ldots, x_n)=\sum_{\beta\in B}\prod_{i=1}^{n}\left(\frac{1}{2}(1+\beta_i)(d(x_i))^{|\beta_i|}+\frac{1}{2}(1-\beta_i)(p_i-1-d(x_i))^{|\beta_i|}\right).$$

Proof. Determine the number of vertices $(y_1, \ldots, y_n)$ adjacent to $(x_1, \ldots, x_n)$ according to a fixed n-tuple $\beta=(\beta_1, \ldots, \beta_n)\in B$. The number of possibilities of choosing y_i depends on β_i. If $\beta_i=1$ we can take for y_i anyone of $d(x_i)$ vertices adjacent to x_i in G_i. For $\beta_i=0$ we have only one possibility $(y_i=x_i)$ and for $\beta_i=-1$ we can choose anyone of $p_i-1-d(x)$ vertices not adjacent to x_i in G_i. For all these three possibilities the numbers have a unique analytic expression (the expression behind the product symbol in the last formula). Taking the sum over all β's we get the total vertex degree.

This completes the proof.

Theorem 1. *The number q of edges in $GDP(B;\, G_1, \ldots, G_n)$ is given by*

$$2q=\sum_{\beta\in B}\prod_{i=1}^{n}\left((\beta_i^2+\beta_i)\,q_i^{|\beta_i|}+(1-\beta_i^2)\,p_i+(\beta_i^2-\beta_i)\left(\binom{p_i}{2}-q_i\right)^{|\beta_i|}\right),$$

where p_i and q_i are the numbers of vertices and edges in G_i $(i=1, \ldots, n)$.

Proof. Taking the sum of vertex degrees $d(x_1, \ldots, x_n)$, determined by Lemma 1. over all vertices $(x_1, \ldots, x_n)$ and exchanging the order of summation we get the theorem.

Theorem 2. *If $A_1, \ldots, A_n$ are the adjacency matrices of graphs $G_1, \ldots, G_n$, then $G=GDP(B;\, G_1, \ldots, G_n)$ has the adjacency matrix*

$$A=\sum_{\beta\in B}\boxed{\bigotimes_{i=1}^{n}}\left(\frac{1}{2}(1+\beta_i)A_i^{|\beta_i|}+\frac{1}{2}(1-\beta_i)\bar{A}_i^{\,|\beta_i|}\right), \tag{1}$$

where $\beta=(\beta_1, \ldots, \beta_n)$, $\bar{A}_i$ is the adjacency matrix of the complement $\bar{G}_i$ of G and $\boxed{\otimes}$ denotes the Kronecker multiplication of matrices.

P r o o f. Let in every of graphs $G_1, \ldots, G_n$ the vertices be ordered (labelled). We shall give lexicographic order to the vertices of G (which represent the ordered n-tuples of vertices of graphs $G_1, \ldots, G_n$) and form the adjacency matrix A according to this ordering. Then we have according to (1)

$$a \stackrel{\text{def}}{=} (A)_{(x_1, \ldots, x_n)(y_1, \ldots, y_n)} = \sum_{\beta \in B} (M_{1\beta})_{x_1 y_1} \cdots (M_{n\beta})_{x_n y_n},$$

where

$$M_{i\beta} = \frac{1}{2}(1+\beta_i) A_i^{|\beta_i|} + \frac{1}{2}(1-\beta_i) \overline{A}_i^{|\beta_i|}.$$

Using the definition of *GDP* we see that $a=1$ if $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent, otherwise $a=0$.

This completes the proof.

T h e o r e m 3. *Let $G_1, \ldots, G_n$ be regular graphs of degrees $r_1, \ldots, r_n$, respectively. Let the graph G_i $(i=1, \ldots, n)$ have p_i vertices, adjacency matrix A_i and the spectrum consisting of eigenvalues λ_{ij_i} $(j_i=1, \ldots, p_i)$, where $\lambda_{i1}=r_i$. Let further $\overline{\lambda}_{i1}=p_i-1-\lambda_{i1}$, $\overline{\lambda}_{ij_i}=-1-\lambda_{ij_i}$ $(j_i=2, \ldots, p_i)$. Then $GDP(B; G_1, \ldots, G_n)$ has the spectrum consisting of eigenvalues $\Lambda_{j_1, \ldots, j_n}$ $(j_i=1, \ldots, p_i;\ i=1, \ldots, n)$, where*

$$\Lambda_{j_1, \ldots, j_n} = \sum_{\beta \in B} \prod_{i=1}^{n} \left(\frac{1}{2}(1+\beta_i) \lambda_{ij_i}^{|\beta_i|} + \frac{1}{2}(1-\beta_i) \overline{\lambda}_{ij_i}^{|\beta_i|} \right).$$

The eigenvector $u_{j_1, \ldots, j_n} = u_{1j_1} \otimes \cdots \otimes u_{nj_n}$ corresponds to the eigenvalue $\Lambda_{j_1, \ldots, j_n}$, where u_{ij_i} is the eigenvector belonging to λ_{ij_i} in G_i.

P r o o f. Using expression for the adjacency matrix A of $GPD(B; G_1, \ldots, G_n)$ we can immediately check the relation $A u_{j_1, \ldots, j_n} = \Lambda_{j_1, \ldots, j_n} u_{j_1, \ldots, j_n}$, which proves the theorem.

Theorems 2 and 3 represent generalizations of particular result form [1] and [2], which are obtained by the very some technique. Note that Theorem 3 holds for arbitrary graphs if the *GDP* is the *NEPS*.

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