PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 22 (36), 1977, pp. 259—265

STRONGLY ALMOST CONVERGENT SEQUENCES

J. Swetits

(Received June 26, 1976)

1. Introduction

C. Chou [4] has provided a characterization of the set of multipliers of the space of almost convergent sequences. Subsequently, Duran [5] characterized the multipliers as the intersection of the multipliers of the convergence domains of all positive, strongly regular matrices. An open question is whether or not the set of multipliers of the space of almost convergent sequences is the bounded convergence domain of any conservative summability matrix. This question is answered in the negative. The result is obtained as a consequence of the relationship between the multipliers and the important class of wedge spaces introduced by Bennett [2].

2. Preliminaries

ω denotes the space of all complex valued sequences, topologized by means of coordinatewise convergence. A vector subspace of ω is called a sequence space. A sequence space E, with a locally convex topology, τ, is a K space provided that the inclusion mapping $(E, \tau) \to \omega$ is continuous. In addition, if τ is complete and metrizable, then (E, τ) is an FK space. A normed FK space is a BK space.

The following spaces will be used in the sequel:

m, the space of bounded sequences;
c, the space of convergent sequences;
c_0, the space of null sequences;
ac, the space of almost convergent sequences;
ac_0, the space of sequences that are almost convergent to 0.

Each of the above is a BK space when topologized by means of the norm

$$\|x\| = \sup_n |x_n|.$$

The space of almost convergent sequences was introduced by Lorentz [8].

Let e^j denote the sequence $(0, \ldots, 0, 1, 0, \ldots)$ with a one in the j^{th} position; Φ the linear span of the e^j's; m_0 the collection of sequences of zeros and ones; and $sp(m_0)$ the linear span of m_0.

A sequence space E is called solid (monotone) if the coordinatewise product $xy \in E$ whenever $x \in m$ $(x \in sp(m_0))$ and $y \in E$.

If (E, τ) is a K space containing Φ and $e^j \to 0$ in τ, then (E, τ) is called a wedge space [2].

$x \in E$ is said to have AK if $P_n(x)$ converges to x in τ, where

$$P_n(x) = \sum_{j=0}^{n} x_j e^j.$$

If each $x \in E$ has AK, then (E, τ) is an AK space.

If E, F is a separated dual pair of vector spaces, then $\sigma(E, F)$ denotes the weak topology on E by F, and $\tau(E, F)$ denotes the Mackey topology on E by F (see, e. g., [12]).

If $A = (a_{nk})$ is an infinite matrix of complex numbers, Ax denotes the sequence defined ay

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \qquad n = 0, 1, 2, \ldots .$$

The convergence domain of A, written (A), is $\{x \in \omega : Ax \in c\}$.

The bounded convergence domain of A is $m \cap (A)$. (A) can be topologized in such a way that it is a separable FK space [1, p. 199].

Definition 2.1, *$x \in \omega$ is called strongly almost convergent to L if $\{|x_j - L|\}$ is almost convergent to* 0.

sac denotes the set of strongly almost convergent sequences. sac_0 denotes the set of sequences that are strongly almost convergent to 0.

It's easy to show that *sac* is a proper closed linear subspace of *ac*. Furthermore, *sac* contains c, and sac_0 is solid.

3. The space sac

The following definition is taken from [11].

Definition 3.1: *A set of positive integers, J, has τ-density zero if the characteristic function of J is almost convergent to* 0.

Chou's result may be stated as

Theorem 3.2 [4]. *$x \in \omega$ is a multiplier of ac if and only if there exists a complex number L such that, for each $\varepsilon > 0$, $\{n : |x_n - L| \geq \varepsilon\}$ has τ-density zero.*

From Theorem 3.2 and definition 2.1, it is easy to see that x is a multiplier of *ac* if and only if $x \in sac$.

Proposition 3.3. *$sp(m_0 \cap ac_0)$ is dense in sac_0.*

Proof. Let $x \in sac_0$ and let $\varepsilon > 0$ be given. Let $J = \{n : |x_n| \geq \varepsilon\}$. Define $y = \{y_k\}$ by $y_k = x_{n_k}$, $n_k \in J$. Then $y \in m$.

Choose $z \in sp(m_0)$ such that

$$\sup_k |y_k - z_k| < \varepsilon.$$

This is possible since $sp(m_0)$ is dense in m. Define $w = \{w_n\} \in sp(m_0 \cap ac_0)$ by

$$w_n = \begin{cases} z_k, & n = n_k \in J \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sup_n |x_n - w_n| \leqslant \varepsilon.$$

Q. E. D.

Proposition 3.3 naturally raises the question: Is $sp(m_0 \cap ac_0)$ barrelled in sac_0? The answer is no. It is a consequence of Lorentz's work on summability function [9, p. 312] that, if $x \in sp(m_0 \cap ac_0)$, then

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n} x_j}{\sqrt{n+1}} = 0.$$

Define the matrix $A = (a_{nk})$ by

$$a_{nk} = \begin{cases} \frac{1}{\sqrt{n+1}} & 0 \leqslant k \leqslant n \\ 0 & k > n. \end{cases}$$

(A) includes $sp(m_0 \cap ac_0)$, but not c_0 and, hence not sac_0. Since (A) is an FK space, it follows from [3, Theorem 1] that $sp(m_0 \cap ac_0)$ is not barrelled in sac_0. We have the following result.

Theorem 3.4. *If E is a separable FK space containing c_0 and $sp(m_0 \cap ac_0)$, then F contains sac_0.*

Before proving Theorem 3.4, we establish a lemma.

l denotes $\{x \in \omega\colon \sum_{k=0}^{\infty} |x_k| < \infty\}$.

Lemma 3.5: *If $\{x^{(n)}\}$ is a sequence in l, then $x^{(n)} \to 0$ in $\sigma(l, c_0 + sp(m_0 \cap ac_0))$ implies that $x^{(n)} \to 0$ in $\sigma(l, sac_0)$.*

Proof: If $x^{(n)} \to 0$ in $\sigma(l, c_0 + sp(m_0 \cap ac_0))$, then $x^{(n)} \to 0$ in $\sigma(l, c_0)$ so that

$$\|x^{(n)}\| = \sup_n \sum_{k=0}^{\infty} |x_k^{(n)}| < \infty.$$

Let $y \in sac_0$ and let $\varepsilon > 0$ be given. From Proposition 3.3; there is $z \in sp(m_0 \cap ac_0)$ such that

$$\sup_k |y_k - z_k| < \varepsilon.$$

Choose N so that $n>N$ implies

$$\left|\sum_{k=0}^{\infty} x_k^{(n)} z_k\right| < \varepsilon.$$

Then, if $n>N$,

$$\left|\sum_{k=0}^{\infty} x_k^{(n)} y_k\right| \leqslant \left|\sum_{k=0}^{\infty} x_k^{(n)}(y_k - z_k)\right| + \left|\sum_{k=0}^{\infty} x_k^{(n)} z_k\right| < \varepsilon \|x^{(n)}\| + \varepsilon.$$

Thus, $x^{(n)} \to 0$ in $\sigma(l, sac_0)$.

Q. E. D.

Proof of Theorem 3.4: $c_0 + sp(m_0 \cap ac_0)$ is a monotone sequence space. Thus l is $\sigma(l, c_0 + sp(m_0 \cap ac_0))$ sequentially complete [2, p. 55]. It follows from [3, Theorem 5] that the inclusion mapping

$$(c_0 + sp(m_0 \cap ac_0),\ \tau(c_0 + sp(m_0 \cap ac_0), l)) \to E$$

is continuous.

sac_0 also has the property that l is $\sigma(l, sac_0)$ sequentially complete. Thus, from Lemma 3.5, $\sigma(l, sac_0)$ and $\sigma(l, c_0 + sp(m_0 \cap ac_0))$ defines the same convergent sequences and, hence, the same Cauchy sequences. It follows that the two topologies define the same compact sets [7, p. 1010]. Thus the topology $\tau(c_0 + sp(m_0 \cap ac_0), l)$ is the restriction of $\tau(sac_0, l)$ to $c_0 + sp(m_0 \cap ac_0)$.

Let $x \in sac_0$. Since $(sac_0, \tau(sac_0, l))$ is an AK space [2, p. 54] $\{P_n(x)\}$ is Cauchy in $\tau(sac_0, l)$. It follows that $\{P_n(x)\}$ is Cauchy in $(c_0 + sp(m_0 \cap ac_0), \tau(c_0 + sp(m_0 \cap ac_0), l))$. Thus $\{P_n(x)\}$ is Cauchy in E. Since E is a complete K space, $\{P_n(x)\}$ must converge in E to x.

Q. E. D.

4. Relation to wedge spaces

Following Bennett [2, p. 51], we let $r = \{r_n\}$ denote a non-decreasing, unbounded sequence of positive integers with $r_0 = 1$. If $x \in \omega$, let $c_n(x)$ denote the number of non-zero terms in $\{x_0, \ldots, x_n\}$ for each $n = 0, 1, 2, \ldots$.

If E is a sequence space, we write

$$E^\alpha = \{y \in \omega : \sum_{n=0}^{\infty} |x_n y_n| < \infty \ \forall x \in E\}.$$

Define

$$(m_0 \cap ac_0, r) = \{x \in m_0 \cap ac_0 : c_n(x) \leqslant r_n, \quad n = 0, 1, \ldots\}.$$

Note that $sp(m_0 \cap ac_0, r)$ is a monotone sequence space. Denote $sp(m_0 \cap ac_0, r)$ by $M_0(r)$.

Lemma 4.1. *$(M_0(r), \tau(M_0(r), M_0^\alpha(r)))$ is a wedge space.*

Proof. The proof is similar to that of Theorem 2 of [2].

Suppose that $(M_0(r), \tau(M_0(r), M_0^\alpha(r)))$ is not a wedge space. There is a neighborhood of zero, U, and an infinite set J of positive integers such that

$$e^j \notin U, \quad j \in J..$$

From J, select an infinite subset, J_1, of τ-density zero such that for each positive integer n, the number of elements of J_1 between 0 and n is less than or equal to r_n.

Define $x \in M_0(r)$ by

$$x_j = \begin{cases} 1 & j \in J_1 \\ 0 & j \notin J_1. \end{cases}$$

Since $(M_0(r), \tau(M_0(r), M_0^\alpha(r)))$ is an AK space [2, p. 54], $P_n(x) \to x$. This implies $e^j \to 0$, $(j \to \infty, j \in J_1)$ which is a contradiction.

Q. E. D.

Theorem 4.2. *If E is a separable FK space containing $M_0(r)$ for some $r = \{r_n\}$, then E is a wedge space.*

Proof. The inclusion mapping

$$(M_0(r), \tau(M_0(r), M_0^\alpha(r))) \to E$$

is continuous because $M_0^\alpha(r)$ is $\sigma(M_0^\alpha(r), M_0(r))$ sequentially, complete ([2, p. 55] and [3, Theorem 5]). The result now follows from Lemma 4.1.

Q. E. D.

Remarks. It is a consequence of [2, Theorem 1] that any wedge FK space must contain $M_0(r)$ for some r. Thus Theorem 4.2 characterizes separable wedge FK spaces.

As a consequence of Theorem 4.2, we obtain the result that *sac* is not the bounded convergence domain of any conservative summability matrix.

Corollary 4.3. *If $\{A_n\}_{n=1}^\infty$ is a countable collection of matrices such that each (A_n) includes sac_0, then there is a bounded, not almost convergent, sequence which is limited by every A_n.*

Proof. Since (A_n) is a separable FK space, the hypothesis and Theorem 4.2 imply that each (A_n) is a wedge space. The result now follows from [2, p. 54 and p. 60].

Q. E. D.

Theorem 4.2 also yields the following improvement to [2, Corollary to Theorem 3].

Corollary 4.4. *If E is a separable FK space and F is a wedge FK space such that E contains $F \cap sac_0$ then E is a wedge space.*

5. Matrix transformations on sac

In this section we characterize those matrices which map *sac* to *sac*.

Theorem 5.1. *Let $A = (a_{nj})$ be an infinite matrix of complex numbers. $Ax \in sac$ whenever $x \in sac$ if and only if*

(i) $\|A\| = \sup_n \sum_{j=0}^{\infty} |a_{nj}| < \infty$;

(ii) *For each $j = 0, 1, 2, \ldots,$ there exists a_j such that $\{a_{nj} - a_j\}_{n=0}^\infty \in sac_0$;*

(iii) $\sum_{j=0}^{\infty} |a_j| < \infty$;

(iv) $\left\{\sum_{j=0}^{\infty} a_{nj}\right\}_{n=0}^{\infty} \in sac$; *and*

(v) *Fox each set J of positive integers of τ density zero*

$\{\sum_{j\in J} a_{nj} - a_j\}_{n=0}^{\infty} \in sac_0$.

P r o o f. (*Necessity*) Condition (i) follows because A maps c_0 to m. Condition (ii) and (iv) are obvious, [10, Theorem 1.3.2].

To establish the necessity of (iii), we note that [8, p. 198]

$$|a_j| \leqslant \sup_n |a_{nj}|.$$

Thus, for each $k = 0, 1, 2, \ldots$

$$\sum_{j=0}^{k} |a_j| \leqslant \sup_n \sum_{j=0}^{k} |a_{nj}| \leqslant \|A\|.$$

Let $J = \{j_0, j_1, \ldots\}$ be an infinite set of positive integers of τ density zero. Define the matrix $B = (b_{nk})$ by

$$b_{nk} = a_{n, j_k}.$$

B maps m to sac and, hence, B maps m to ac. By a results of Duran [6, p. 77], for each $x \in m$ the almost convergent limit of Bx is $\sum_{k=0}^{\infty} a_{j_k} x_k$. Define $y \in sac_0$ by

$$y_j = \begin{cases} x_k & j = j_k \\ 0 & \text{otherwise.} \end{cases}$$

It follows that Ay is strongly almost convergent to $\sum_{k=0}^{\infty} a_{j_k} x_k$, thus establishing the necessity of (v).

(*Sufficiency*): Let $x \in sac_0$. Let $\varepsilon > 0$ be given and choose $z \in sp(m_0 \cap ac_0)$ such that

$$\sup_k |x_k - z_k| < \varepsilon.$$

Let $z = \sum_{i=0}^{r} b_i y^{(i)}$, where each b_i is a scalar and each $y^{(i)} \in m_0 \cap ac_0$. For each $i = 0, 1, \ldots, r$, let J_i be the set of positive integers of τ density zero such that

$$y_j^{(i)} = \begin{cases} 1 & j \in J_i \\ 0 & j \notin J_i. \end{cases}$$

Then

$$\left|\sum_{j=0}^{\infty} (a_{nj} - a_j) x_j\right| \leqslant 2\|A\|\varepsilon + \sum_{i=0}^{r} |b_i| \left|\sum_{j\in J_i} a_{nj} - a_j\right|$$

From Condition (v), there exists a positive integer P such that, if $p>P$,

$$\frac{1}{p+1}\sum_{q=0}^{p}\left|\sum_{j=0}^{\infty}(a_{n+q,j}-a_j)x_j\right|<2\|A\|\varepsilon+\varepsilon\sum_{i=0}^{r}|b_i|$$

uniformly in $n=0, 1, 2, \ldots$. Thus A maps sac_0 to sac.

The proof is completed by observing that, if $y\in sac$, then $y=Le+x$ where $x\in sac_0$, L is the strong almost convergent limit of y, and $e=\{1, 1, \ldots, 1, \ldots\}$.

Q. E. D.

Remark. The proof of Theorem 5.1 shows that, if $x\in sac_0$, then Ax is strongly almost convergent to $\sum_{j=0}^{\infty} a_j x_j$.

REFERENCES

[1] G. Bennett, *A representation theorem for summability domains*, Proc. London. Math. Soc. 24 (1972), 193—203.

[2] G. Bennett, *A new class of sequence spaces with applications in summability theory*, J. reine, u. angew. Math. 266 (1974), 49—75.

[3] G. Bennett and N. J. Kalton, *Inclusion theorems for K spaces*, Canad. J. Math. 25 (1973), 511—524.

[4] C. Chou, *The multipliers of the space of almost convergent sequences*, Illinois J. Math. 16 (1972), 687—694.

[5] J. P. Duran, *Strongly regular matrices, almost convergence, and Banach limits*, Duke Math. J. 39 (1972), 497—502.

[6] J. P. Duran, *Infinite matrices and almost convergence*, Math. Zeit, 128 (1972), 75—83.

[7] D. J. H. Garling, *On topological sequences spaces*, Proc. Camb. Phil. Soc. 63 (1967), 997—1019.

[8] G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta. Math. 80 (1948), 167—190.

[9] G. G. Lorentz, *Direct theorems on methods of summability*, Canad. J. Math. 1 (1949), 305—319.

[10] G. M. Peterson, *Regular matrix transformations*, McGraw-Hill, New York, 1966.

[11] R. A. Raimi, *Convergence, density, and τ-density of bounded sequences*, Proc. Amer. Math. Soc. 14 (1963), 708—712.

[12] A. P. Robertson and W. J. Robertson, *Topological vector spaces*, Cambridge 1973.

Dr. J. Swetits
Department of Mathematical and Computing Sciences
Old Dominion University
Norfolk, Virginia 23508