PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 22 (36), 1977, pp. 255—258

PERMUTATION GROUP EXTENSION OF GROUPOIDS

Lee Sin-Min

(Received October 3. 1975)

Abstract. Given a groupoid X and a permutation group G on X, a new groupoid from $X \cup G$ is constructed. A necessary and sufficient conditions for the resulting groupoid to be a semigroup and a sufficient condition for this groupoid to be subdirectly irreducible are given.

1. Introduction

A universal algebra of type $\langle 2 \rangle$ is called a gıoupoid [4]. It is called subdirectly irreducible if its lattice of congruences has a unique minimal congruence relation. In particular, all simple groupoids i. e. those groupoids whose congruence lattices are isomorphic to the two elements chain are subdirectly irreducible.

The class of subdirectly irreducible groupoids attracts the attention of algebraists because every groupoid is a subdirect product of subdirectly irreducible groupoids. One would like to be able to classify all possible subdirectly irreducible groupoids up to isomorphism. Such a classification is, however, impossible. For it was shown by Lee S. M. and Sein-Aye [3] that every groupoid is a subgroupoid of a simple groupoid.

The main object of this note is to introduce a method of constıuction new groupoids from old ones and by this way to construct a large class of subdirectly irreducible groupoids.

2. The construction

Let X be a set and S_X be the symmetric group on X. A permutation group is a triple (X, G, i) where X is a set, G is an abstract group and $i: G \to S_X$ is a homomorphism. In pratice we identify G with its image in S_X,

Now assume X is a groupoid and (X, G, i) is a permutation group. The small letters a, b, c will always denote elements of X. The capitals $A, B. C$ will always denote elements of G. Denote the multiplication in X and G by juxtaposition. Now we define a binary operation * on $I(X, G, i) = X \cup G$ as follows:

$$x * y = \begin{cases} xy & \text{if } x, y \in X \text{ or } x, y \in G \\ x & \text{if } x \in X \text{ and } y \in G \\ x(y) & \text{if } x \in G \text{ and } y \in X. \end{cases}$$

where $x(y)$ means the image of y under the map $i(x) \in S_X$.

Then $\langle I(X, G, i), * \rangle$ is a groupoid with X as its ideal. We call this groupoid as the permutation group extension of X by G.

Note that if X is a semigroup, the resulting system $I(X, G, i)$ need not be a semigroup.

E x a m p l e 1. Let $X = \{a, b, c\}$ be a semilattice with two non comparable elements a and b. Let $G = \langle \mathbf{Z}/2\mathbf{Z}, + \rangle$ and let $i_1, i_2 : G \to S_X$ be defined by

$$i_1(0) = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} \quad i_1(1) = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} \quad i_2(0) = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} \quad i_2(1) = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$$

The multiplication tables of $I(X, G, i_k)$, $k = 1, 2$; are given as follows:

$*_1$	a	b	c	0	1
a	a	c	c	a	a
b	c	b	c	b	b
c	c	c	c	c	c
0	a	b	c	0	1
1	a	b	c	1	0

$*_2$	a	b	c	0	1
a	a	c	c	a	a
b	c	b	c	b	b
c	c	c	c	c	c
0	a	b	c	0	1
1	b	a	c	1	0

$I(X, G, i_1)$ is a semigroup, however $I(X, G, i_2)$ is not a semigroup for

$$(1 *_2 a) *_2 b = b *_2 b = b \quad \text{and} \quad 1 *_2 (a *_2 b) = 1 *_2 c = c.$$

Recall a map f from a groupoid X to itself is called a left translation if $f(x \cdot y) = f(x) \cdot y$. We shall denote the set of all left translations by $\mathcal{T}_1(X)$.

We have the following characterization of $I(X, G, i)$ to be a semigroup.

T h e o r e m 1. *The permutation group extension of a groupoid X by G is a semigroup if and only if*

(1) *X is a semigroup,*

(2) *$i(G) \subset \mathcal{T}_1(X)$ and*

(3) *$x(i(A)(y)) = xy$ for all x, y in X and A in G.*

P r o o f. Suppose $I(X, G, i)$ is a semigroup then (1) is clearly hold. Let $A \in G$ and $x, y \in X$ then we have

(i) $$(x * A) * y = x * (A * y)$$

and

(ii) $$(A * x) * y = A * (x * y).$$

(i) implies condition (3) and (ii) shows that $i(A)$ is a left translation of X for each A in G.

The proof of the other direction of the statement involves the verification of the associativity of $*$ in eight different cases: XXX, XGX, GXG, ..., depending on where three arbitrarily chosen elements lie in $I(X, G, i)$. That associativity holds in all cases follows directly from conditions (1), (2) and (3).

In particular we have

Example 2. Let X be any semigroup and $G=\{id_X\}$ consists only the identity of X then clearly $I(X, G)$ is a semigroup. This construction is the well-known "adjoint a unity to a semigroup" procedure [2].

Corollary 1. *Let X be a null semigroup, i.e. a semigroup with zero* 0 *such that $xy=0$ for all x, y in X. The set of all left translations is $\mathcal{T}_1(X)=\{f: X\to X: f(0)=0\}$. If G is a permutation group of X such that $i(G)\subset\mathcal{T}_1(X)$ then $I(X, G, i)$ is a semigroup.*

Corollary 2. *Let X be a left absorption semigroup, i.e. semigroup satisfies the identity $xy=x$ for all x, y in X. The set of left translations is $\mathcal{T}_1(X)=\{f: X\to X\}$. For any permutation group G of X, the permutation group extension of X is a semigroup.*

3. A sufficient condition for $I(X, G, i)$ to be subdirectly irreducible

We recall that a permutation group G of X is said to act n-transitively on X if for any two sets of n elements in X, say $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ there exists a permutation A in G such that $A(a_i)=b_i$.

The main result of this section is the following.

Theorem 2. *Let X be a groupoid and (X, G, i) be a permutation group such that i is an imbedding. If $i(G)$ acts 2-transitively on X then the groupoid $I(X, G, i)$ is subdirectly irreducible. Furthermore its congruence lattice is isomorphic to the lattice obtained from the lattice of all normal subgroups of G by adjoining a null and a maximal element.*

Proof. Let $\theta^*=X\times X\cup\{(A, A): A\in G\}$, then it is clear that it is a congruence relation of $I(X, G, i)$. We claim that θ^* is in fact the minimal congruence relation of $I(X, G, i)$.

Suppose θ is a non- trivial congruence relation on $I(X, G, i)$. Then there exists two distinct elements x and y in $I(X, G. i)$ such that $x\theta y$. Consider the following all possible cases:

Case 1. $x, y\in X$.

Since G acts 2-transitively on X, therefore for any u, v in X there exists A in G such that $A(x)=u$ and $A(y)=v$. Thus left multiplication $x\theta y$ both sides by A we have $u\theta v$. Hence $\theta^*\subset\theta$.

Case 2 $x\in X$ and $y\in G$.

Since G acts 2-transitively on X there is always an $A \in G$ such that $A(x) \neq x^2$. Then right multiplication $x \theta y$ both sides by $y^{-1} A$ we obtatn $x \theta A$ and hence $x^2 \theta A(x)$ which also reduces to case 1.

Case 3 $x, y \in G$.

Since $x \neq y$ in G thus there exists $s \in X$ such that $x(s) \neq y(s)$. Then right multiplication both sides of $x \theta y$ by s we obtain $x(s) \theta\ y(s)$ which reduces to case 1.

In any case we have $\theta^* \subset \theta$ therefore θ^* is the unique minimal congruence relation and hence $I(X, G, i)$ is subdirectly irreducible.

In fact all the non trivial congruence relation of $I(X, G, i)$ is of the form $\theta(N) = X \times X \cup \equiv_N$ where $\equiv_N$ is the congruence relation of G induced by the normal subgroup N.

In general, the lattice of congruence of a groupoid is not a modular lattice. it is interesting to note that the above construction of subdirectly irreducible groupoids always give modular congruence lattice.

Example 3. Let $\langle F, +, . \rangle$ be a field. Let G be the set of affine trasformation of F, i. e. maps of the form $u \mapsto mu + c$, m, $c \in F$, $m \neq 0$ then G is a group which acts exactly 2-transitively on F. The groupoids $I((F, +), G)$ and $I((F, .), G)$ are not isomorphic however they have the isomorphic congruence lattices.

Example 4. Let X be any groupoid with cardinality $|X| = n \geqslant 4$.

Let A_n be the alternating group of the set X. It is well-known that A_n acts $(n-2)$-transitively on X and A_n is a simple group except when $n = 4$.

Thus the congruence lattice of $I(X, A_n)$ looks like the following lattice:

if $n = 4$

if $n \geqslant 5$

Corollary 3. *Every groupoid is an ideal of some subdirectly irreducible groupoids.*

REFERENCES

[1] R. H. Buck, *A survey of binary systems*, Springer, Berlin 1958.

[2] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. I, Math. Surveys, № 7, Amer. Math. Society, 1961.

[3] Lee Sin-Min and Sein-Aye, *On simple groupoids*, Nanta Mathematica Vol. 8, 1975. p. 30—33.

[4] A. I. Malcev, *Algebraic systems*, Springer, 1973.

Département de Mathématiques
Université de Paris-Sud, Centre d'Orsay
91405 Orsay, France.