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ON A GENERALIZATION OF HUMBERT POLYNOMIALS

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1. Introduction: Gould [3] has defined generalized Humbert polynomials $P_n(m, x, y, p, C)$ as

$$(1.1) \qquad (C - m\,x\,t + y\,t^m)^p = \sum_{n=0}^{\infty} t^n P_n(m, x, y, p, C).$$

Legendre polynomials are defined as [5]

$$(1.2) \qquad (1 - 2\,xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$$

giving

$$(1.3) \qquad P_n(x) = \frac{\left(\frac{1}{2}\right)_n (2\,x)^n}{n!}\; {}_2F_1\left[\begin{matrix} -\frac{n}{2}, & \frac{-n+1}{2}; & \frac{1}{x^2} \\ n - \frac{1}{2}; & & \end{matrix}\right].$$

We observe that $(1 - 2\,xt + x^3\,t^2)^{-\frac{1}{2}}$ generates polynomials which are closely associated with $P_n(x)$ since

$$(1.4) \qquad (1 - 2\,xt + x^3\,t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (2\,x)^n}{n!}\; {}_2F_1\left[\begin{matrix} -\frac{n}{2}, & \frac{-n+1}{2}; & x \\ -\frac{1}{2}+n; & & \end{matrix}\right] t^n.$$

This leads us to define a new class of polynomials $P_n^{(l)}(m, x, a, b, \nu)$ by the relation

$$(1.5) \qquad (1 - axt + b\,x^l\,t^m)^{-\nu} = \sum_{n=0}^{\infty} t^n P_n^{(l)}(m, x, a, b, \nu),$$

where a, b, m, ν and l are parameters.

The polynomials defined by (1.5) includes (1.1) and may be regarded as generalization of a number of polynomials viz. Humbert, Gegenbauer etc. and the polynomials closely associated with them of the type (1.4).

In the present note we shall be studying the properties of these polynomials. It is interesting to note that in the study of these polynomials we require the use of the operators of the type $(\mu\, x^{\alpha} D + \eta\, x^{\beta} D)$ and the generalized number $A^{n+1}_{q+1}(a_0; a_1, \ldots, a_n)$ defined by [6]

$$(1.6)\qquad \prod_{\gamma=1}^{n} \mathfrak{D}_{\gamma} f = \sum_{q=0}^{n} A^{n+1}_{q+1}(a_0; a_1, \ldots, a_n)\, x^{a_0+a_1+\cdots+a_n-n+q} \cdot D^q (x^{-a_0} f),$$

where $x^{a_\gamma} D = \mathfrak{D}_\gamma$ and $\prod_{\gamma=1}^{n} \mathfrak{D}_\gamma = \mathfrak{D}_n \cdot \mathfrak{D}_{n-1} \cdots \mathfrak{D}_1$ and a's are parameters.

Obviously (1.6) generalizes Stirling numbers $A^{\alpha}_{n,k,i}$ given by Chak [1] by the relation

$$(1.7)\qquad (x^k D)^n \cdot f = \sum_{i=0}^{n} A^{\alpha}_{n,k,i}\, x^{nk-n+i+\alpha} \cdot D^i (x^{-\alpha} \cdot f),$$

since, for $a_1 = \cdots = a_n = k$ and $a_0 = \alpha$, (1.6) reduces to (1.7)

We shall also study the properties of the above numbers and polynomials associated with them in the note [6].

2. *Hypergeometric Expression for* $P^{(l)}_n(m, x, a, b, \nu)$.: When $m > 1$, we obtain

$$(2.1)\qquad (1 - axt + bx^l t^m)^{-\nu} = \sum_{k=0}^{\infty} \frac{(\nu)_k a^k t^k x^k}{k!} \cdot {}_mF_{m-1}\left[\begin{matrix} -\frac{k}{m}, \frac{-k+1}{m}, \ldots, \frac{-k+m-1}{m}; \\ \frac{-\nu-k+1}{m-1}, \ldots, \frac{-\nu-k+m-1}{m-1}; \end{matrix} \frac{bm^m a^{-m}}{(m-1)^{m-1}} x^{l-m}\right]$$

and thus

$$(2.2)\qquad P^{(l)}_n(m, x\; a, b, \nu) = \frac{(\nu)_n a^n x^n}{n!}\, {}_mF_{m-1}\left[\begin{matrix} -\frac{n}{m}, \ldots, \frac{-n+m-1}{m} ; \\ \frac{-\nu-n+1}{m-1}, \ldots, \frac{-\nu-n+m-1}{m-1}; \end{matrix} \frac{b\left(\frac{m}{a}\right)^m x^{l-m}}{(m-1)^{m-1}}\right].$$

Here we distinguish between two classes, viz. $l > m$ and $l < m$. If λ be a positive integer, we define for $l = m + \lambda$

$$(2.3)\qquad P^{(l)}_n(m, x, a, b, \nu) = \Phi^{(\lambda)}_n(m, x, a, b, \nu) = \Phi^{(\lambda)}_n(x)$$

and for $l = m - \lambda$

$$(2.4)\qquad P^{(l)}_n(m, x, a, b, \nu) = \Psi^{(\lambda)}_n(m, x, a, b, \nu) = \Psi^{(\lambda)}_n(x).$$

From (2.3) and (2.4) we obtain the relation

$$(2.5)\qquad x^{2n}\Psi^{(\lambda)}_n\left(m, \frac{1}{x}, a, b, \nu\right) = \Phi^{(\lambda)}_n(m, x, a, b, \nu).$$

$\Psi_n^{(\lambda)}(x)$ includes $P_n(m, x, y, p, C)$ and its particular cases like Legendre and Humbert polynomials, where as $\Phi_n^{(\lambda)}(x)$ gives a class of polynomials closely associated wit $\Psi_n^{(\lambda)}(x)$.

3. *Recurrence and other relations.*: We obtain the following recurrence relations for $P_n^{(l)}(m, x, a, b, \nu)$:

(3.1) $$n P_n - (n+\nu-1)\, ax\, P_{n-1} + b\, x^l (n-m+\nu m)\, P_{n-m} = 0;\ n \geqslant m-1,$$

(3.2) $$P'_n - ax\, P'_{n-1} + b\, x^l\, P'_{n-m} = \nu\, a\, P_{n-1} - b\, \nu\, l\, x^{l-1} P_{n-m},$$

(3.3) $$an\, P_n - ax\, P'_n = bl\, x^{l-1}(n-m+1) P_{n+m-1} - bx^l\, m\, P'_{n-m+1};\ n \geqslant m-1,$$

(3.4) $$n P_n + b x^l [\nu (m-1) - (l-1)(n-m)]\, P_{n-m} = x P'_n + b(1-m)\, x^{l+1} P_{n-m},$$

and

(3.5) $$x^l \nu (m-1)\, b\, P^{(l)}_{n-m}(m, x, a, b, \nu+1) = x P'_n - n P_n =$$

$$= x^l \nu (m-1) \sum_{i_1+i_2+\cdots+i_1+\frac{i}{\nu}} P_{i_1} \cdot P_{i_2} \cdots P_{i_1+\frac{i}{\nu}}$$

where $\left(1+\dfrac{1}{\nu}\right)$ is a positive integer.

For $m=2$, a differential equation of the second degree is given as

(3.6) $$(4\, b\, x^l - a^2 x^2)\, D^2 \omega - [2(2n-1)\, b\, l\, x^{l-1} + a^2 \{\nu (2-l) - n(l-1) + 1 - n\} \cdot x] \cdot D\omega - [bn\, l\, (l-n\, l-2)\, x^{l-2} + a^2 n \{n(l-1) - \nu (2-l)\}]\, \omega = 0.$$

Further we easily obtain that

(3.7) $$\Phi_n^{(\lambda)}(x) = \frac{(\nu)_n}{(2n)!}\, a^n D^n x^{2n}\, {}_lF_{l-1} =$$

$$= \left[\begin{matrix} -\dfrac{n}{m}, \dfrac{-n+1}{m}, \ldots, \dfrac{-n+m-1}{m}, \dfrac{n+\lambda}{\lambda}, \ldots, \dfrac{n+\lambda}{\lambda}, \\ \dfrac{-\nu-u+1}{m-1}, \ldots, \dfrac{-\nu-n+m-1}{m-1}, \dfrac{2n+1}{\lambda}, \ldots, \dfrac{2n+\lambda}{\lambda}, \end{matrix} \; \frac{b\left(\dfrac{m}{a}\right)^m x^{\lambda}}{(m-1)^{m-1}}\right]$$

and

(3.8) $$\Psi_n^{(\lambda)}(x) = \frac{(\nu)_n}{(2n)!}\, a^n \cdot D^n \cdot x^{2n} \cdot {}_{m+\lambda}F_{m+\lambda-1} =$$

$$= \left[\begin{matrix} -\dfrac{n}{m}, \dfrac{-n+1}{m}, \ldots, \dfrac{-n+m-1}{m}, -\dfrac{n+1}{\lambda}, \ldots, -\dfrac{n+\lambda}{\lambda}; \\ -\dfrac{\nu-n+1}{m-1}, \ldots, -\dfrac{\nu-n+m-1}{m-1}, -\dfrac{2n+1}{\lambda}, \ldots, \dfrac{2n+\lambda}{\lambda}; \end{matrix} \; \frac{b\left(\dfrac{m}{a}\right)^m x^{-\lambda}}{(m-1)^{m-1}}\right].$$

4. *Operational Relations.* Let

$$H = H(x, t) = (1 - a x t + x^l t^m)^{-\nu},$$

we obtain

(4.1) $$(-at + l b x^{l-1} t^m) D_t \cdot H = (-ax + m b t^{m-1} x^l) D_x H,$$

and successive operation yields

(4.2) $$(-at D_t + l b x^{l-1} t^m D_t)^n H = (-ax D_x + mbt^{m-1} x^l D_x)^n \cdot H.$$

The operator on the right hand side is of the form

$$(\mu x^\alpha D + \eta x^\beta D) \text{ and}$$

$$(\mu x^\alpha D + \eta x^\beta D)^n = \sum_{s=0}^{n} \mu^{n-s} \eta^s \sum (x^{a_n} D)(x^{a_{n-1}} D) \cdots (x^{a_1} D),$$

where $a_1, a_2, \ldots, a_{n-1}, a_n$ are either α or β, α is taken $(n-s)$ times and β is taken s times. The inner sum is for such combinations of α and β, which has $\binom{n}{s}$ terms.

Now using (4.3) and (1.6), we obtain from (4.2)

$$(-at D_t + l b x^{l-1} t^m D_t)^n f(x, t) =$$

$$= \sum_{s=0}^{n} (-a)^{n-s} x^{s(l-1)} (m b t^{m-1})^s \sum \sum_{q=1}^{n} A_{q+1}^{n+1} 0; a_1, \ldots a_n) \cdot x^q \cdot D^q f(x, t)$$

$$= \sum_{q=1}^{n} \sum_{s=0}^{n} [\sum A_{q+1}^{n+1} 0; a_1, \ldots, a_n)] (-1)^{n-s} a^{m-s} m^s b^s t^{s(m-1)} \cdot x^{q+s(l-1)} \cdot D^q f(x, t).$$

The sum $\sum A_{q+1}^{n+1} (0; a_1, \ldots, a_n)$ has the same conditions as that of (4.3) with $\alpha = 1$ and $\beta = l$.

Now put

(4.4) $$Q_q^{(n, l)} (m, x, a, b, t)$$

$$= \sum_{s=0}^{n} [A_{q+1}^{n+1} (0; a_1, \ldots, a_n)] (-1)^s a^{n-s} (mb)^s t^{(m-1)s} x^{q+(l-1)s}.$$

Particular case of (4.4) are Stirling numbers of the second kind, because for $l = 0$, $m = 1$, $a = 1$, and $b = 1$, we get

(4.5) $$Q_q^{(n, 0)} (1, x, 1, 1, t) = (x-1)^q S(n, q) = (x-1)^q \frac{(-1)^q}{q!} \sum_{j=0}^{q} (-1)^j \binom{q}{j} j^n;$$

and evidently (4.4) is the extention of $Q_j^\gamma (m, x, y, t)$ given by Gould [3].

(4.6) $$(t D_t)^\gamma f(x, t) = \sum_{j=1}^{\gamma} Q_j^\gamma (m, x, y, t) D_x^j f(x, t).$$

5. *Derivatives of* $P_n^{(l)}(m, x, a, b, \nu)$.: Let us consider the operator $\Phi(x)D$, where

(5.1) $$\Phi(x)=\sum_{\gamma} a_{\gamma}\, x^{\lambda\gamma}.$$

Further let

(5.2) $$\Phi_m(x)=\sum_{\gamma} a_{\gamma}^{(m)}\, x^{\gamma\lambda}=\Phi_m,$$

then we find from (1.6) that

$$(\Phi_n(x)D)\cdots(\Phi_1(x)D)\Phi_0(x)$$

$$=\sum_{i_n}\sum_{i_{n-1}}\cdots\sum_{i_1} a_{i_n}^{(n)}\, a_{i_{n-1}}^{(n-1)}\cdots a_{i_1}^{(1)}\sum_{q=0}^{n} A_{q+1}^{n+1}(\lambda a_0;\lambda i_1,\ldots,\lambda i_n)\cdot$$

$$\cdot\, x^{q+\lambda(i_1+i_2+\cdots+i_n+a_0)-n}\cdot D^q(x^{-a_0\lambda}\Phi_0).$$

For convenience let $\Phi_0(x)=\sum_{i_0} a_{i_0}^{(0)} x^{\lambda i_0}$. Then

$$(\Phi_n D)(\Phi_{n-1}D)\cdots\Phi_1(D)\Phi_0$$

$$=\sum_{i_n}\sum_{i_{n-1}}\cdots\sum_{i_1}\sum_{i_0} a_{i_n}^{(n)}\cdots a_{i_1}^{(1)}\, a_{i_0}^{(0)}\, A_1^{n+1}(\lambda i_0;\ \lambda i_1,\ldots,\lambda i_n)\cdot x^{\lambda(i+i_1+\cdots+i_n)-n}$$

and using the relation [6]

$$A_1^{n+1}(a_0; a_1, a_2,\ldots,a_n)=(a_0+a_1+\cdots+a_{n-1}-n+1)\,A_1^n(a_0; a_1,\ldots,a_{n-1})$$

we get

(5.3) $$(\Phi_n D)\cdots(\Phi, D)\Phi_0$$

$$=\sum_{i_n}\sum_{i_{n-1}}\cdots\sum_{i_1}\sum_{i_0} a_{i_n}^{(n)}\, a_{i_{n-1}}^{(-1)}\cdots a_{i_1}^{(1)},\ a_{i_0}^{(0)}\, x^{\lambda(i_0+\cdots+i_n)-n}\{\lambda i_0\}^{(n-1,\lambda i_{n-1})}$$

where $\{\alpha\}^{(n-1,a_{n-1})}=\alpha(\alpha+a_1-1)(\alpha+a_1+a_2-2)\cdots(\alpha+a_1+\cdots+a_{n-1}-n+1)$.

Again for any polynomial α (x) in x, we have the rule of differentiation

(5.4) $$D^n(\alpha^{-\nu})=\sum_{k=0}^{n-1}(-1)^{n-k}(\nu)_{n-k}\,\alpha^{\alpha-\nu-n+k}\,B_{k+1}^n,$$

where

(5.5) $$B_{k+1}^n=\sum_{i_0+i_1+\cdots+i_k=n-k}[(D\alpha)^{i_k}D]\,[(D\alpha)^{i_{k-1}}D]\cdots[(D\alpha)^{i_1}D]\,[(D\alpha)^{i_0}].$$

The rule is verified by the method of induction. Now let $\alpha=1-axt+$ $+bx^l t^m$ and $H=\alpha^{-\nu}$.

We obtain from (5.3), (5.4) and (5.5)

(5.6) $$D^nH=\sum_{k=0}^{n-1}(-1)^{n-k}(\nu)_{n-k}\,H^{1+\frac{n-k}{\nu}}\,B_{k+1}^n,$$

where

(5.7) $$B_{k+1}^n=(-at)^{n-k}\sum_{i_0+\cdots+i_k=n-k}\ \sum_{j_k=0}^{i_k}\cdots\sum_{j_0=0}^{i_0}\binom{i_k}{j_k}\cdots\binom{i_0}{j_0}\cdot\left(\frac{-b\,l\,t^{m-1}}{a}\right)^{i_0+\cdots+j_k}\cdot$$

$$\cdot\, x^{(l-1)(j_0+\cdots+j_k)-k}\cdot\{(l-1)j_0\}^{(k-1,(l-1)j_{k-1})}.$$

It is immediately seen that, when $l=0$, we obtain the result [3]

(5.8) $$D^n \cdot H = (at)^n (\nu)_n H^{1+\frac{n}{\nu}}.$$

Now from (1.5) we obtain

(5.9) $$D^k H = t^k \sum_{n=0}^{\infty} t^n D^k P^l_{n+k} (m, x, a, b, \nu); \quad l<m$$

$$= t^k \sum_{n=0}^{\infty} t^n D^k \Psi^{(\lambda)}_{n+k} (m, x, a, b, \nu),$$

and

(5.10) $$D^k H = \sum_{n=0}^{\infty} t^n D^k P^{(l)}_n (m, x, a, b, \nu); \quad l>m$$

$$= \sum_{n=0}^{\infty} t^n D^k \Phi^{(\lambda)}_n (m, x, a, b, \nu).$$

Thus from (5.6) and (5.7), we obtain

(5.11) $$D^n \Phi^{(\lambda)}_{\gamma} (m, x, a, b, \nu)$$
$$= \sum_{k=0}^{n-1} (-1)^{n-k} (\nu)_{n-k} B^n_{k+1} \sum \Phi^{(\lambda)}_{\omega_1} \Phi^{(\lambda)}_{\omega_2} \cdots \Phi^{(\lambda)}_{\omega_{1+\frac{\nu}{n}}}$$

where $w_1 + w_2 + \omega_3 + \cdots + \omega_{1+\frac{\nu}{n}} = \gamma - \{n-k+(m-1)\sum j_k\}$,

and $1+\dfrac{\nu}{n}$ is an integer; and

(5.12) $$D^n \Psi^{(\lambda)}_{n+p} (m, x, a, b, \nu)$$
$$= \sum_{k=0}^{n-1} (-1)^{n-k} (\nu)_{n-k} \; B^n_{k+1} \sum \Psi^{(\lambda)}_{\omega_1} \cdots \Psi^{(\lambda)}_{\omega_{1+\frac{\nu}{n}}}$$

where $\omega_1 + \omega_2 + \cdots + \omega_{1+\frac{\nu}{n}} = p+k-(m-1)\sum j_k$.

These results are obvious extensions of Gould [3] results and hence that of Đokovic [2].

Since

(5.13) $$D^n \Phi^{(\lambda)}_n (m, x, a, b, \nu) = \frac{(\nu)_n a^n}{n!} {}_lF_{l-1}$$

$$= \left[\begin{matrix} -\dfrac{n}{m}, \dfrac{-n+1}{m}, \ldots, \dfrac{-n+m-1}{m}, \dfrac{n+1}{l-m}, \ldots, \dfrac{n+l-m}{l-m}, \\ \dfrac{-\nu-n+1}{m-1}, \ldots, \dfrac{-\nu-n+m-1}{m-1}, \dfrac{1}{l-m}, \ldots, \dfrac{l-m-1}{l-m}, \end{matrix} \; \frac{b\left(\dfrac{m}{a}\right)^m x^{\lambda}}{(m-1)^{m-1}} \right],$$

we obtain from (5.11)

$$(5.14)\qquad \frac{(\nu)_n a^n}{n!}\, {}_lF_{l-1}$$

$$= \left[\begin{matrix} -\frac{n}{m}, \frac{-n+1}{m}, \ldots, \frac{-n+m-1}{m}, \frac{n+1}{l-m}, \ldots, \frac{n+l-m}{l-m}, \\ \frac{-\nu-n-1}{m-1}, \ldots, \frac{-\nu-n+1-m}{m-1}, \frac{1}{l-m}, \ldots, \frac{l-m-1}{l-m}, \end{matrix} \quad \frac{b\left(\frac{m}{a}\right)^m x^\lambda}{(m-1)^{m-1}}\right]$$

$$= \sum_{k=0}^{n-1} (-1)^{n-k} (\nu)_{n-k} B^n_{k+1} \sum \Phi^{(\lambda)}_{\omega_1} \ldots \Phi^{(\lambda)}_{\omega_{1+\frac{\nu}{n}}}$$

where $\omega_1 + \omega_2 + \cdots + \omega_{1+\frac{\nu}{n}} = k - (m-1) \sum j_k$

and

$$(5.15)\qquad D^n \Psi^{(\lambda)}_n (m, x, a, b, \nu) = (\nu)_n a^n.$$

Thus from (5.12) and (5.15) we obtain

$$(5.16)\qquad \frac{D^n \Psi^{(\lambda)}_{n+p} (m, x, a, b, \nu)}{D^n \Psi^{(\lambda)}_n (m, x, a, b, \nu)}$$

$$= [(\nu)_n a^n]^{-1} \sum_{k=0}^{n-1} (-1)^{n-k} (\nu)_{n-k} B^n_{k+1} \sum \Psi^{(\lambda)}_{\omega_1} \ldots \Psi^{(\lambda)}_{\omega_{1+\frac{\nu}{n}}}$$

where $\omega_1 + \omega_2 + \cdots + \omega_{1+\frac{\nu}{n}} = k - (m-1) \sum j_k$.

Also we observe that

$$(5.17)\qquad \sum_{p=0}^{\infty} t^p D^n \Psi^{(m)}_{n+p} (m, x, a, b, \nu)$$

$$= a^n \binom{-\nu}{n} (1 - axt + bt^m)^{-\nu-n},$$

thus

$$(5.18)\qquad D^n \Psi^{(m)}_{n+p} (m, x, a, b, \nu) = a^n \binom{-\nu}{n} \Psi^{(m)}_p (m, x, a, b, \nu+n)$$

which is the same as that for (1.1) [3]

$$(5.19)\qquad D^n P_{n+s} (m, x, y, p, b) = (-m)^n n! \binom{p}{n} P_s (m, x, y, p-n, b)$$

and generalizes the familiar formula [8, p. 330] and [7, 47.14] for Gegenbauer polynomials $C^{(\nu)}_n (x)$

$$(5.20)\qquad D\, C^{(\nu)}_n (x) = 2\nu\, C^{(\nu+1)}_{n-1} (x).$$

6. *Some other Polynomials Related to* $P_n^{(l)}$: From (2.2) we have

$$P_n^{(l)}(m, x, a, b, \nu) = \sum_{\gamma=0}^{[n/m]} \frac{(\nu)_{n-\gamma(m-1)}\, a^{n-\gamma m}(-b)^{\gamma}\, x^{n+\gamma(l-m)}}{\gamma!\,(n-\gamma m)!}. \tag{6.1}$$

Gould-Hopper [4] have defined generalized Hermite polynomials $g_n^{(\gamma)}(x, h)$ as

$$g_n^{(\gamma)}(x, h) = \sum_{k=0}^{[n/\gamma]} \frac{n!}{k!\,(n-\gamma k)!}\, h^k x^{n-k\gamma} \tag{6.2}$$

and Zeitlin [9] has defined $R_n(x, m, q)$ polynomials as

$$R_n(x, m\ q) = \frac{n}{m} \sum_{k=0}^{[n/m]} (-1)^k \frac{[(n-1)-(m-1)k]!\ q^{(m-2)k/2}\,(2x)^{n-km}}{(n-mk)!}. \tag{6.3}$$

We observe the following limiting cases:

$$\underset{\nu\to\infty}{L_t}\ \nu^{-\frac{n}{m}}\, \Psi_n^{(m)}\left(m, \frac{x}{\nu^{\frac{m-1}{m}}},\ a, b, \nu\right) = \frac{a^n}{n!}\, g_n^{(m)}(x, -ba^m), \tag{6.4}$$

which gives an extension to the well known limit

$$\underset{\lambda\to\infty}{L_t}\ \lambda^{-\frac{n}{2}}\, C_n^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right) = \frac{2^{n/2}}{n!}\, H_n(x/\sqrt{2}), \tag{6.5}$$

and

$$\underset{\nu\to\infty}{L_t}\ \nu^{-\frac{n}{m}}\, P_n^{(l)}\left(m, \frac{x}{\nu^{\frac{m-1}{m}}},\ a, b, \nu^{l\left(\frac{m-1}{m}\right)}\right)$$

$$= \frac{a^n}{n!} \sum_{k=0}^{[n/m]} \frac{n!\,(-ba^m)^k}{k!\,(n-mk)!}\, x^{n+kl-mk},$$

which leads us to define

$$g_n^{(m)}(x, l, h) = \sum_{k=0}^{[n/m]} \frac{n!\, h^k\, x^{n+kl-mk}}{k!\,(n-mk)!}. \tag{6.6}$$

Thus

$$\underset{\nu\to\infty}{L_t}\ \nu^{-n/m}\, P_n^{(l)}\left(m, \frac{x}{\nu^{\frac{m-1}{m}}},\ a\ b, \nu^{l\left(\frac{m-1}{m}\right)}\right) = \frac{a^n}{n!}\, g_n^m(x, l, -ba^m),$$

and

$$g_n^{(m)}(x, 0, h) = g_n^{(m)}(x, h).$$

Also

$$\sum_{n=0}^{\infty} g_n^{(m)}(x, l, h)\, t^n = e^{tx+hx^l t^m}. \tag{6.7}$$

Thus we see that (6.6) generalizes (6.2).
Again

$$\underset{\nu\to 0}{L_t}\ \frac{P_n^{(l)}(m, x, a, b, \nu)}{\nu} = \sum_{k=0}^{[n/m]} \frac{[n-1-(m-1)k]!\ a^{n-mk}(-b)^k\, x^{n+k(l-m)}}{k!\,(n-mk)!}.$$

Hence we define

$$(6.8)\qquad R_n^{(l)}(x, m, q) = \frac{n}{m} \sum_{k=0}^{[n/m]} \frac{[(n-1)-(m-1)k]!\, q^{\frac{m-2}{2}k}\, (2x)^{n+k(l-m)}}{k!\,(n-mk)!}.$$

Thus

$$\underset{\nu=0}{L_t} \frac{P_n^{(l)}(m, x, a, b, \nu)}{\nu} \quad a^n \binom{m}{n} R_n^{(l)}\left(\frac{x}{2}, m, (b\, a^{-m})^{2/m-2}\right),$$

and

$$R_n^{(0)}(x, m, q) = R_n(x, m, q).$$

Also

$$(6.9)\qquad \sum_{n=0}^{\infty} t^n R_n^{(l)}(x, m, q) = \frac{1 - \frac{2(1-m)}{m} xt}{1 - 2xt + q^{\frac{m-2}{2}} x^{l(m-2)} \cdot t^m}.$$

Thus we see that (6.8) generalizes (6.3).

The inverse series relations for $P_n^{(l)}(m, x, a, b, \nu)$ are

$$(6.10)\qquad \binom{-\nu}{n}(-ax)^n = \sum_{k=0}^{\infty} (-1)^k \binom{-\nu-n+k}{k} \frac{-\nu+mk-n}{-\nu+k-n} \cdot b^k x^{lk} \cdot$$

$$\cdot P_{n-km}^{(l)}(m, x\, a, b, \nu),$$

and

$$(6.11)\qquad P_n^{(l)}(m, x, a, b, \nu) = \sum_{k=0}^{[n/m]} \binom{-\nu-n+m+k}{k} \binom{-\nu}{n-mk} b^k a^{-mk} \cdot x^{n-mk+lk}.$$

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