

ON SETS UNDER CERTAIN TRANSFORMATIONS IN R_N

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For every ω belonging to a metric space Ω , there is a certain transformation T_ω , transforming a measurable set in R_N (N -dimensional Euclidean space) into a measurable set in R_N . The measure considered in this paper is the Lebesgue measure. Let the transformations T_ω satisfy the following conditions:

(I) There exists $\omega_0 \in \Omega$ such that for every sphere $K = S[a, r] \subset R_N$ and every sequence $\{\omega_n\} (\omega_n \in \Omega)$ converging to ω_0 ,

$$\lim_{n \rightarrow \infty} [\sup \{ |a - T\omega_n(K)| \}] = r \text{ holds.}$$

(II) If E and F be measurable sets in R_N such that $F \subset E$, then for every $\omega \in \Omega$, $T_\omega(F) \subset T_\omega(E)$.

(III) If E be a measurable set in R_N and $\omega_n \rightarrow \omega_0$, then

$$\lim_{n \rightarrow \infty} |T\omega_n(E)| = |T\omega_0(E)| = |E|,$$

where $|E|$ denotes the measure of the set E .

Then in a recent paper, among other results M. Pal [4] has proved the following theorem which extends a theorem (Theorem 1.1) of T. Neubrunn and T. Šalát [3].

Theorem. *Let $T\omega_n (\omega_n \in \Omega)$ be the transformations satisfying the conditions (I), (II) and (III) and the sequence $\{\omega_n\}$ converge to ω_0 (in Ω). Let A be a set of positive measure in R_N . Then there exists a natural number N_0 such that for $n \geq N_0$, $A \cap T\omega_n(A)$ is a set of positive measure.*

In this paper we study some properties of sets in R_N under transformation like T_ω which transforms a measurable set in R_N into a measurable set in R_N . In the first section by introducing some conditions in R_N we extend the result of Theorem 1 of [4] and derive as a particular case the well-known result that the difference set $D(A, B)$ of two sets A and B of positive measure in R_N contains a sphere [1, 6]. In the second section we introduce the definitions of *null symmetric sets* and *category symmetric sets* and extend the idea of 'equivalent sets' as introduced by K. C. Ray in [5]. In the last theorem (Theorem 8) we prove a fundamental result on the sequence of transformations $\{T\omega_n\}$.

Notations: (a) $S[c, \rho]$ stands for the closed ball with centre c and radius ρ , while $S(c, \rho)$ denotes the open ball with the same centre and radius.

(b) Difference set $D(A, B)$ means the set of all vectors $x - y$, where $x \in A$ and $y \in B$ (A and B are two sets in R_N).

(c) $|x|$ denotes the norm of the vector x .

(d) \bar{A} denotes the closure of the set A .

(e) A/B denotes the set of all those vectors of the set A which do not belong to the set B .

(f) For $a \in R_N, A \subset R_N$, the symbol $\{|a - A|\}$ denotes the set of all numbers $|a - x|$, where $x \in A$.

We also note a well-known result [2] that if $E_1, E_2 \dots$ be a sequence of measurable sets and $E = \bigcap_{k=1}^{\infty} E_k$ such that $E \supset E_2 \supset E_3 \supset \dots$, then $|E| = \lim_{n \rightarrow \infty} [|E_n|]$.

1. Suppose that there is a certain transformation $T\omega$ which transforms a measurable set in R_N into another measurable set in R_N for every ω belonging to a metric space Ω .

Let the transformations $T\omega$ satisfy the following conditions:

(i) There exists $\omega_0 \in \Omega$ such such that for any two spheres $K_1 = S[a, r_1]$ and $K_2 = S[b, r_2]$ in R_N and every sequence $\omega_n \in \Omega$ converging to ω_0 ,

$$\begin{aligned} \lim_{n \rightarrow \infty} [\sup \{|a - T\omega_n(K_2)|\}] &= \min(r_1, r_2) \text{ if } r_1 \neq r_2 \\ &= r \text{ if } r_1 = r_2 = r \text{ (say)}. \end{aligned}$$

(ii) If E and F be two measurable sets in R_N such that $F \subset E$, then for every $\omega \in \Omega, T\omega(F) \subset T\omega(E)$.

(iii) If E be a measurable set in R_N and $\omega_n \rightarrow \omega_0$ (in Ω),

$$\lim_{n \rightarrow \infty} |T\omega_n(E)| = |T\omega_0(E)| = |E|.$$

If $K_1 = K_2$, then condition (i) becomes condition (I) of [4]. The conditions (ii) and (iii) are same as conditions (II) and (III) respectively of [4].

Theorem 1. *Let $\omega_n (\omega_n \in \Omega)$ be a sequence converging to $\omega_0 (\omega_0 \in \Omega)$ and $T\omega_n$ be the transformations satisfying the conditions (i), (ii) and (iii). Let A and B be two sets of positive measure in R_N . Then there exists a natural number N_0 , such that for $n \geq N_0, A \cap T\omega_n(B)$ is a set of positive measure.*

Proof. Since A and B are two sets of positive measure in R_N , by Density Theorem, there exist spheres $K_1 = S[a, r_1]$ and $K_2 = S[b, r_2]$ ($r_2 < r_1$) such that $|K_1/A| < \varepsilon |K_1|$ and $|K_2/B| < \varepsilon |K_2|$, where $0 < \varepsilon < \frac{p-1}{p(p^N+1)}$, p being any positive number for which $pr_2 \geq r_1$.

Let $|K_1| = \delta$ so that $|K_2| \geq \delta/p^N$. Also let $\sup \{|a - T\omega_n(K_2)|\} = d_n$.

Then from the condition (i) there exists a natural number N_1 such that for $n \geq N_1$, $|d_n - r_2| < r_1 - r_2$. So for $n \geq N_1$, $T\omega_n(K_2) \subset K_1$ i.e., $T\omega_n(K_2 \cap B) \subset K_1$. Again by the condition (iii) there exists a natural number N_2 such that for $n \geq N_2$,

$$\|T\omega_n(K_2 \cap B) - |K_2 \cap B|\| < \delta/p^{N+1}.$$

Let $N_0 = \max(N_1, N_2)$. Also let $C = K_1 \cap A$ and $C_n = T\omega_n(K_2 \cap B)$.

Now for $n \geq N_0$, $C \cap C_n = K_1 - [C' \cup C_n]$, where dashes denote the complements with respect to K_1 . Then for $n \geq N_0$,

$$\begin{aligned} |C \cap C_n| &\geq |K_1| - [|C'| + |C_n|] \\ &= |K_1| - [|K_1/A| + |K_1| - |T\omega_n(K_2 \cap B)|] \\ &= |T\omega_n(K_2 \cap B)| - |K_1/A|, \\ &> |K_2 \cap B| - \frac{\delta}{p^{N+1}} - |K_1/A| \\ &= |K_2| - |K_2/B| - |K_1/A| - \frac{\delta}{p^{N+1}} \\ &> |K_2| - \varepsilon |K_2| - \varepsilon |K_1| - \frac{\delta}{p^{N+1}} \\ &\geq \frac{\delta}{p^N} - \varepsilon \frac{\delta}{p^N} - \varepsilon \cdot \delta - \frac{\delta}{p^{N+1}} \\ &= \left(\frac{\delta}{p^N} - \frac{\delta}{p^{N+1}}\right) - \varepsilon \left(\delta + \frac{\delta}{p^N}\right) \\ &= \delta \left[\frac{p-1}{p^{N+1}} - \varepsilon \left(1 + \frac{1}{p^N}\right)\right] \\ &= \delta \left[\frac{p-1}{p^{N+1}} - \varepsilon \left(\frac{p^N+1}{p^N}\right)\right] \\ &> 0, \text{ since } 0 < \varepsilon < \frac{p-1}{p(p^N+1)}. \end{aligned}$$

Thus for $n \geq N_0$, $(K_1 \cap A) \cap T\omega_n(K_2 \cap B)$ is a set of positive measure. Hence by applying condition (ii) we obtain that $A \cap T\omega_n(B)$ is a set of positive measure in R_N for $n \geq N_0$.

This completes the proof.

Corollary 1. *If in Theorem 1 we put $A=B$ and choose $a=b$ in (i) then Theorem 1 of [4] follows.*

That $D(A, B)$ contains a sphere, as shown by S. Kurepa [1] and K. C. Ray [6], will appear as a particular case of Theorem 1 in the next corollary.

Corollary 2. *If A and B be two sets of positive measure in R_N , then the difference set $D(A, B)$ contains a sphere.*

Proof. Let $b - a = c$ and $\{\omega_n\}$ ($\omega_n \in R_N$) converge to c ($\in R_N$).

If we put $g\omega_n(x) = x + \omega_n$, where $x \in R_N$ then evidently $g\omega_n$ satisfies the conditions (i), (ii) and (iii) in R_N . Then by the above theorem there exists a natural number N_0 such that for every $n \geq N_0$, there is a vector $x_n \in B$ for which $x_n + \omega_n \in A$. Let $|\omega_{N_0} - c| = \eta$. Then for every $\omega \in S[c, \eta]$ there is a vector $x \in B$ such that $x + \omega \in A$. Thus the difference set $D(A, B)$ contains a sphere $S[c, \eta]$, $\eta > 0$.

This proves the corollary.

Theorem 2. *Let the transformations $T\omega_n$ ($\omega_n \in \Omega$ and $\{\omega_n\} \rightarrow \omega_0 \in \Omega$, $\omega_n \neq \omega_{n+1}$, $n = 1, 2, \dots$) satisfy the conditions (i), (ii) and (iii) in R_N . Let A and B be two sets of positive measure in R_N and p be any positive integer, then there exist mutually distinct $\omega_1, \omega_2, \dots, \omega_p$ in Ω such that*

$$A \cap T\omega_1(B) \cap T\omega_2(B) \cap \dots \cap T\omega_p(B),$$

is a set of positive measure.

Proof. Let $\{\omega'_n\}$ ($\omega'_n \in \Omega$) be a sequence converging to ω_0 ($\in \Omega$). Since A and B are two sets of positive measure, by Theorem 1 there exists a natural number N_1 such that for $n \geq N_1$, $A \cap T\omega'_n(B)$ is a set of positive measure. We choose a natural number $n_1 \geq N_1$ and put $\omega'_1 = \omega_1$. Then $A \cap T\omega_1(B) = C_1$, say, is a set of positive measure, and $C_1 \subset A$.

Next let $\{\omega''_n\}$ ($\omega''_n \in \Omega$) be a sequence converging to ω_0 ($\in \Omega$). Since $|C_1| > 0$, by Theorem 1 there is a natural number N_2 such that for $n \geq N_2$, $C_1 \cap T\omega''_n(B)$ is a set of positive measure. Let n_2 be such that $n_2 \geq N_2$ and $\omega''_{n_2} = \omega_2$ ($\neq \omega_1$). Then $C_2 = C_1 \cap T\omega_2(B) = \{A \cap T\omega_1(B)\} \cap T\omega_2(B)$ is a set of positive measure and $C_2 \subset C_1 \subset A$. Continuing this process after finite number of steps we obtain a set C_{p-1} where $|C_{p-1}| > 0$ and a transformation $T\omega_p$ ($\omega_1 \neq \omega_2 \neq \dots \neq \omega_p$) such that

$$A \supset C_1 \supset C_2 \supset \dots \supset C_{p-1};$$

and $C_p = C_{p-1} \cap T\omega_p(B)$ is a set of positive measure, i.e.,

$$A \cap T\omega_1(B) \cap \dots \cap T\omega_p(B),$$

is a set of positive measure.

This completes the proof.

Theorem 3. *Let A, B_1, B_2, \dots, B_m ($m > 1$) be bounded sets in R_N having positive measure. Let $T\omega_n$ ($\omega_n \in \Omega$) satisfy the conditions (i), (ii) and (iii) in R_N . Then there exist sequences $\{\omega_n^m\}$ converging to ω_0^m and a positive integer N_0 such that for $n \geq N_0$,*

$$A \cap T_n^1(B_1) \cap T_n^2(B_2) \cap \dots \cap T_n^m(B_m),$$

is a set of positive measure.

Proof. Let $\{\omega_n^1\} (\omega_n^1 \in \Omega)$ be a sequence converging to $\omega_0^1 (\in \Omega)$. Since A and B_1 are two sets of positive measure, by Theorem 1, there exists a natural number N_1 such that for $n \geq N_1$, $A \cap T\omega_n^1(B_1)$ is a set of positive measure. Let $C_1 = A \cap T\omega_n^1(B_1)$ for any value of $n \geq N_1$. Then $|C_1| > 0$ and $C_1 \subset A$. Next, let $\{\omega_n^2\} (\omega_n^2 \in \Omega)$ be a sequence converging to $\omega_0^2 \in \Omega$. Since $|C_1| > 0$ and $|B_2| > 0$, again by Theorem 1, there exists a natural number N_2 such that for $n \geq N_2$, $C_1 \cap T\omega_n^2(B_2)$ is a set of positive measure. Let $C_1 \cap T\omega_n^2(B_2) = C_2$ for any value of $n \geq N_2$. Then $|C_2| > 0$ and $C_2 \subset C_1 \subset A$. Thus for any value of $n \geq \max(N_1, N_2)$, $A \cap T\omega_n^1(B_1) \cap T\omega_n^2(B_2)$ is a set of positive measure.

Proceeding in this way, we obtain a set C_{m-1} , where $|C_{m-1}| > 0$ and $A \supset C_1 \supset C_2 \supset \dots \supset C_{m-1}$, and a sequence $\{\omega_n^m\} \rightarrow \omega_0^m$ such that $C_{m-1} \cap T\omega_n^m(B_m)$ is a set of positive measure for any $n \geq N_m$. If we consider $N_0 = \max(N_1, N_2, \dots, N_m)$, then for $n \geq N_0$,

$$A \cap T\omega_n^1(B_1) \cap T\omega_n^2(B_2) \cap \dots \cap T\omega_n^m(B_m)$$

is a set of positive measure.

This completes the proof.

Now we shall prove a theorem in which the transformations $T\omega_n (\omega_n \in \Omega)$ satisfy the following condition (iv) in addition to (i), (ii) and (iii).

(iv) If $\{\omega_n\} (\omega_n \in \Omega)$ be a sequence converging to ω_0 (in Ω) and if B be a closed set, then $T\omega_n(B)$ is closed for sufficiently large n .

Theorem 4. Let the transformations $T\omega_n (\omega_n \in \Omega)$ and $\{\omega_n\} \rightarrow \omega_0 (\in \Omega)$, ($\omega_n \neq \omega_{n+1}$, $n = 1, 2, \dots$) satisfy the conditions (i), (ii), (iii) and (iv) in R_N . Let A and B be two closed sets of positive measure in R_N . Then there exist $\{\omega_k\}$, $\omega_k \in \Omega$ and a vector $\xi \in R_N$ such that $\xi \in A \cap T\omega_k(B)$, $k = 1, 2, \dots$.

Proof. Proceeding indefinitely as in the proof of Theorem 2 we obtain, by repeated application of Theorem 1, a sequence of closed sets $\{C_n\}$ such that

$$A \supset C_1 \supset C_2 \supset \dots \supset C_{n-1} \supset C_n \supset \dots$$

and a sequence $\{\omega_k\} (\omega_k \in \Omega)$ $\omega_i \neq \omega_j (i \neq j)$ such that $C_k = C_{k-1} \cap T\omega_k(B)$,

$$|C_k| > 0, \bar{C}_k = C_k \text{ and } C_k \subset C_{k-1}, k = 1, 2, \dots$$

So there exists a vector $\xi \in A$ such that $\xi \in T\omega_k(B)$ $k = 1, 2, \dots$.

This proves the theorem.

2. In this part we prove some theorems (Theorem 6 and Theorem 7) extending some results on "equivalent sets" of [5]. In doing so in place of translation we apply transformation $T\omega$ as already defined.

Theorem 5. Let the transformations $T\omega_n (\omega_n \in \Omega)$ satisfy the condition (i) in R_N , where the sequence $\{\omega_n\} (\omega_n \in \Omega)$ converges to $\omega_0 (\in \Omega)$. Also let $K_1 = S(a, r_1)$ and $K_2 = S(b, r_2)$ be two spheres in R_N . If ξ be any vector belonging to the sphere K_2 , then $T\omega_n(\xi) \in K_1$ for sufficiently large n .

Proof. Let ξ be any vector belonging to $K_2 = S(b, r_2)$. Since K_1 and K_2 are both open, we consider a closed sphere $S_1 = S[a_1, \gamma_1] \subset K_1$ and a closed sphere $S_2 = S[b_1, \gamma_2] \subset K_2$ containing the point ξ ($\gamma_2 < \gamma_1$). Then by the condition (i) $T\omega_n(\xi) \in S_1 \subset K_1$ for sufficiently large value of n .

This proves the theorem.

We now introduce the following definitions:

Definition 1. Let A and B be two sets in R_N and $\omega_n (\in \Omega)$ converge to $\omega_0 (\in \Omega)$. Let X and Y be any null subsets of A and B respectively. If for every vector $\xi \in A/X$, there exists $T\omega_n$ such that $T\omega_n(\xi) \in B$, for sufficiently large n , and also if for every vector $\eta \in B/Y$, there exists $T\omega_n$ such that $T\omega_n(\eta) \in A$ for sufficiently large n we say that the sets A and B are *null symmetric*.

Definition 2. In Definition 1 if X and Y are sets of the first category, then the sets A and B are said to be *category symmetric*.

Now in proving our next theorem we require the following condition:

(v) If $|E| = 0$, then $|T\omega_n(E)| = 0, n = 1, 2, \dots$ where

$$\{\omega_n\} \rightarrow \omega_0, \omega_n \in \Omega, \omega_0 \in \Omega.$$

Theorem 6. Let the transformations $T\omega_n$ with their inverses satisfy the conditions (i) and (v). If A and B are two sets (in R_N) containing almost all points of the spheres $K_1 = S(a, r_1)$ and $K_2 = S(b, r_2)$ respectively, then A and B are *null symmetric*.

Proof. Let $A = K_1/E_1$ and $B = K_2/E_2$, where E_1 and E_2 are null subsets of K_1 and K_2 respectively, i.e., $|E_1| = |E_2| = 0$. Let

$$Q = K_1 \cap \left(\bigcup_{n=1}^{\infty} T\omega_n^{-1}(E_2) \right).$$

Since

$$\left| \bigcup_{n=1}^{\infty} T\omega_n^{-1}(E_2) \right| \leq \sum_{n=1}^{\infty} |T\omega_n^{-1}(E_2)| = 0,$$

by condition (v), we have $|Q| = 0$. Let $X = Q \cup E_1$. Then obviously $|X| = 0$.

If $\zeta \in A/X$, then $\xi \in X$, i.e., $\xi \in T^{-1}\omega_n(E_2)$, for $n = 1, 2, \dots$, i.e., $T\omega_n(\xi) \in E_2$, for $n = 1, 2, \dots$

Now since K_1 and K_2 are two open spheres by applying condition (i) we have from Theorem 5, $T\omega_n(\xi) \in K_2$ for sufficiently large n . Hence we have $T\omega_n(\xi) \in K_2/E_2$, for sufficiently large n , i.e., $T\omega_n(\xi) \in B$, for sufficiently large n .

By similar consideration we can show that for every vector $\eta \in B/Y$, where $|Y| = 0$, there exist $T\omega_n$ such that $T\omega_n(\eta) \in A$ for sufficiently large n . Accordingly A and B are *null symmetric*.

This proves the theorem.

Corollary. *If $\omega_n \in A$, $\{\omega_n\}$ converging to $\omega_0 \in A$, and $g \omega_n(x) = x + \omega_n$, where $x \in R_N$, then the conditions (i) and (v) are satisfied by $g \omega_n$ with their inverses; and applying Theorem 6, we get Theorem 5 of [5].*

Theorem 7. *Let the transformations $T \omega_n$ with their inverses satisfy the condition (i) and preserve the category, where the sequence $\{\omega_n\}$ ($\omega_n \in \Omega$) converges to ω_0 (in Ω). If A and B be two sets (in R_N) containing all the points of the spheres $K_1 = S(a, r_1)$ and $K_2 = S(b, r_2)$ respectively, with the exception of those belonging to a set of the first category, then A and B are category symmetric.*

Proof. We consider $A = K_1/E_1$ and $B = K_2/E_2$, where E_1 and E_2 are sets of the first category and subsets of K_1 and K_2 respectively.

Then following the same line of argument as in Theorem 6 and replacing the statement " $X = 0$ " by " X is of the first category" we obtain the desired result.

Corollary. *If $g \omega_n(x) = x + \omega_n$, where $x \in R_N$ and $\{\omega_n\}$ ($\omega_n \in A$) be any convergent sequence converging to $\omega_0 (\in A)$, then the conditions of Theorem 7 are satisfied and we get Theorem 6 of [5].*

Theorem 8. *Let the transformations $T \omega_n$ with their inverses satisfy the conditions (i) and (iii), where the sequence $\{\omega_n\}$ ($\omega_n \in \Omega$) converges to ω_0 (in Ω). Let A and B be two sets in R_N containing almost all points of the spheres $K_1 = S(a, r_1)$ and $K_2 = S(b, r_2)$ respectively. Then given $\delta (> 0)$, however small, there exists a set $K_1^\delta \subset K_1$ for which $|K_1^\delta| > |K_1| - \delta$, such that for any vector $\xi \in K_1^\delta$ there exists a subsequence $\{\omega_{n_i}\}$ of $\{\omega_n\}$ for which $T \omega_{n_i}(\xi) \in B$.*

Proof. Let $A_1 = K_1/E_1$ and $B = K_2/E_2$, where E_1 and E_2 are two null subsets of K_1 and K_2 respectively, i.e., $|E_1| = |E_2| = 0$.

Let $\eta_1 + \eta_2 + \eta_3 + \dots + \eta_i + \dots$ be a convergent series of positive terms.

On account of condition (iii), $\lim_{n \rightarrow \infty} |T \omega_n^{-1}(E_2)| = |E_2|$. So for η_1 there exists a natural number n_1 such that

$$||T \omega_{n_1}^{-1}(E_2)| - |E_2|| < \eta_1 \text{ i.e., } |T \omega_{n_1}^{-1}(E_2)| < \eta_1.$$

Similarly for η_2 we can choose a natural number $n_2 > n_1$ such that $|T \omega_{n_2}^{-1}(E_2)| < \eta_2$. In this way we choose a subsequence $\{\omega_{n_i}\}$ of $\{\omega_n\}$ such that corresponding to η_i , $|T \omega_{n_i}^{-1}(E_2)| < \eta_i$ where $n_1 < n_2 < \dots < n_i < \dots$. Let

$$X_i = K_1 \cap \left\{ \bigcup_{k=i}^{\infty} T \omega_{n_k}^{-1}(E_2) \right\},$$

for any value of i , then

$$X_1 \supset X_2 \supset \dots \supset X_i \supset \dots, \text{ and } |X_i| < \sum_{k=i}^{\infty} \eta_k.$$

Now given $\delta (>0)$, however small, we can choose

$$|X_i| < \sum_{k=i}^{\infty} \eta_k < \delta.$$

But

$$|X_i \cup E_1| \leq |X_i| + |E_1| < \sum_{k=i}^{\infty} \eta_k < \delta.$$

Let $K_1^\delta = K_1 - (X_i \cup E_1)$, a subset of K_1 . Then

$$|K_1^\delta| = |K_1| - |X_i \cup E_1|, \text{ i.e. } |K_1^\delta| > |K_1| - \delta.$$

Now let $\xi \in K_1^\delta$. Then $\xi \in K_1 - (X_i \cup E_1)$, i.e., $\xi \in \overline{(X_i \cup E_1)}$ i.e., $\xi \in X_i$, i.e., $\xi \in T\omega_{n_i}^{-1}(E_2)$, i.e., $T\omega_{n_i}(\xi) \in E_2$.

Since K_1 and K_2 are open, by condition (i) it follows that $T\omega_{n_i}(\xi) \in K_2$. Hence we have $T\omega_{n_i}(\xi) \in K_2/E_2$, i.e., for any $\xi \in K_1^\delta$, $T\omega_{n_i}(\xi) \in B$, for all i .

This completes the proof.

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