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# SYSTEMS OF LINEAR BOOLEAN EQUATIONS

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**Abstract.**

In a previous paper [3] we constructed the general solution of a system of linear Boolean equations by using the Löwenhein formula which gives the reproductive general solution of a Boolean equation. In this short communication we consider a slightly more general type of linear Boolean equations and obtain a simpler form of the general solution.

## § 1

The term "linear Boolean equation" has been given two different meanings.

The first one coincides with the classical concept of linear equation applied to the ring structure of a Boolean algebra. Properties specific to systems of linear Boolean-ring equations have been studied by Parker and Bernstein [2] (see also [4] ch. 6 § 3).

The second meaning of linearity in Boolean algebra, suggested by a simple analogy, is due to Löwenheim [1]. Let $(\mathbf{B}, \cup, \cdot, ', 0, 1)$ be an arbitrary Boolean algebra; Löwenheim has studied Boolean equations of the form

$$\bigcup_{j=1}^{n} a_{ij} x_j = b_i \qquad (i := 1, \ldots, m), \tag{1}$$

finding the necessary and sufficient condition

$$b_i \leqslant \bigcup_{j=1}^{n} a_{ij} \prod_{\substack{h=1 \\ h \neq i}}^{m} (a'_{hj} \cup b_h \qquad (i := 1, \ldots, m) \tag{2}$$

for the consistency of (1) and concentrating on the case when the system $\{b_i\}\, i = 1, \ldots, m$ is orthogonal, normal or orthonormal (see also [4] ch. 7 § 3). We have shown [3] that if the system (1) is consistent then it has the reproductive general solution

$$x_j = [\prod_{i=1}^{m} (a'_{ij} \cup b_i)]\, [t_j \cup \bigcup_{i=1}^{m} b_i \prod_{k=1}^{n} (t'_k \cup a'_{ik} \cup \bigcup_{h=1}^{m} a_{hk} b'_h)] \qquad (j := 1, \ldots, n). \tag{3}$$

In the sequel we consider several slight generalizations of the definition (1) of linear equations, by adding a constant term to the left side and by replacing the sign "=" by inequality signs; we prove that all these types of equations and inequations reduce to (1) and we obtain a recurrent form of the general solution to (1),which seems much simpler than (3). An application to the determination of bijectivity domains of Boolen transformations will be given in [5].

We assume that the reader has some acquaintance with Boolean calculus.

## § 2

Consider the following systems of equations and inequations

$$a_i \cup \bigcup_{j=1}^{n} a_{ij} x_j \leqslant b_i \qquad (i:=1, \ldots, m), \tag{4}$$

$$c_i \leqslant a_i \cup \bigcup_{j=1}^{n} a_{ij} x_j \qquad (i:=1, \ldots, m), \tag{5}$$

$$a_i \cup \bigcup_{j=1}^{n} a_{ij} x_j = d_i \qquad (i:=1, \ldots, m), \tag{6}$$

$$\bigcup_{j=1}^{n} a_{ij} x_j \leqslant b_i \qquad (i:=1, \ldots, m), \tag{7}$$

$$c_i \leqslant \bigcup_{j=1}^{n} a_{ij} x_j \qquad (i:=1, \ldots, m). \tag{8}$$

Lemma 1. *The system* (4) *reduces to a system of the form* (1).

Proof. Since $a_i \leqslant b_i$ that is $a_i b_i' = 0$ $(i:=1, \ldots, m)$, is a necessary condition for the consistency of (4), this system can be written

$$\bigcup_{j=1}^{n} a_{ij} b_i' x_j = 0 \qquad (i:=1, \ldots, m). \tag{9}$$

Lemma 2. *The system* (5) *reduces to a system of the form* (8).

Proof. The system (5) can be written

$$c_i a_i' (\bigcup_{j=1}^{n} a_{ij} x_j)' = 0 \qquad (i:=1, \ldots, m)$$

or, equivalently,

$$c_i a_i' \leqslant \bigcup_{j=1}^{n} a_{ij} x_j \qquad (i:=1, \ldots, m). \tag{10}$$

Lemma 3. *The system* (8) *reduces to a system of the form* (1).

Proof. The system (8) can be written

$$\bigcup_{j=1}^{n} a_{ij} x_j = c_i \cup v_i \qquad (i := 1, \ldots, m), \tag{11}$$

where $v_1, \ldots, v_n$ are arbitrary parameters in **B**.

Lemma 4. *The system* (5) *reduces to a system of the form* (1).

Proof. Immediate from lemmas 2 and 3.

Lemma 5. *The system* (6) *reduces to a system of the form* (1).

Proof. Write (6) in the form

$$d_i \leqslant a_i \cup \bigcup_{j=1}^{n} a_{ij} x_j \leqslant d_i \qquad (i := 1, \ldots, m) \tag{12}$$

and apply lemmas 1 and 4, thus obtaining a system of the form (1) with 2m equations.

Lemma 6. *The system* (7) *reduces to a system of the form* (1).

Proof. The system (7) can be written

$$\bigcup_{j=1}^{n} a_{ij} x_j = b_i v_i \qquad (i := 1, \ldots, m), \tag{13}$$

where $v_1, \ldots, v_n$ are arbitrary parameters in **B**.

Proposition 1. *Any system of equations and/or inequations of (some of) the forms* (1), (4)–(8), *reduces to a system of the form* (1).

Proof. Immediate from lemmas 1–6.

Theorem 1. *The vector* $(x_1, \ldots, x_n)$ *is a solution of the system*

$$\bigcup_{j=1}^{n} a_{ij} x_j \leqslant b_i \qquad (i := 1, \ldots, m) \tag{7}$$

*if and only if*

$$x_j \leqslant \prod_{i=1}^{m} (a'_{ij} \cup b_i) \qquad (j := 1, \ldots, n). \tag{14}$$

Proof. The system (7) can be written successively in the following equivalent forms:

$$b_i' \bigcup_{j=1}^{n} a_{ij} x_j = 0 \qquad (i := 1, \ldots, m),$$

$$a_{ij} b_i' x_j = 0 \qquad (i := 1, \ldots, m;\ j := 1, \ldots, n),$$

$$x_j \leqslant a'_{ij} \cup b_i \qquad (i := 1, \ldots, m;\ j := 1, \ldots, n),$$

and this is equivalent to (14).

*to be consistent. Then the vector* $(x_1, \ldots, x_n)$ *is a solution of* (1) *if and only if*

$$(18.j)\qquad \bigcup_{i=1}^{m} b_i \Big(\prod_{k=1}^{j-1} a'_{ik}\Big)\Big[\prod_{k=j+1}^{n} (a'_{ik} \cup x'_k)\Big]$$

$$\leqslant x_j \leqslant \prod_{i=1}^{m} (a'_{ij} \cup b_i) \qquad (j := 1, \ldots, n).$$

Comment. As for theorem 2.

Proof. Write the system (1) in the form

$$(19)\qquad b_i \leqslant \bigcup_{j=1}^{n} a_{ij} x_j \leqslant b_i \qquad (i := 1, \ldots, m)$$

and apply theorems 1 and 2.

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[1] L. Löwenheim, *Gebietdeterminanten.* Math. Ann. 79 (1919), 222—236.

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*to be consistent. Then the vector* $(x_1, \ldots, x_n)$ *is a solution of* (1) *if and only if*

$$(18.j) \qquad \bigcup_{i=1}^{m} b_i \Big(\prod_{k=1}^{j-1} a'_{ik}\Big) \Big[\prod_{k=j+1}^{n} (a'_{ik} \cup x'_k)\Big] \leqslant x_j \leqslant \prod_{i=1}^{m} (a'_{ij} \cup b_i) \qquad (j := 1, \ldots, n).$$

Comment. As for theorem 2.

Proof. Write the system (1) in the form

$$(19) \qquad b_i \leqslant \bigcup_{j=1}^{n} a_{ij} x_j \leqslant b_i \qquad (i := 1, \ldots, m)$$

and apply theorems 1 and 2.

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[1] L. Löwenheim, *Gebietdeterminanten.* Math. Ann. 79 (1919), 222—236.

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