

# $(F, F)$ - AND $(F, F)$ -CONNEXIONS OF AN ALMOST COMPLEX AND AN ALMOST PRODUCT SPACE

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1. *Introduction.* — In an almost complex space an affine connexion is called  $F$ -connexion if the almost complex structure tensor  $F_j^i$  is covariant constant with respect to this connexion. The general  $F$ -connexion has the form ([1], chapter VI, § 2):

$$(1.1) \quad \Gamma_{ji}^k = \overset{\circ}{\Gamma}_{ji}^k - \frac{1}{2} (\overset{\circ}{\nabla}_j F_i^a) F_a^k + \frac{1}{2} (\delta_i^a \delta_b^k - F_i^a F_b^k) W_{ja}^b,$$

where  $W_{ka}^b$  is an arbitrary tensor of the type indicated by the indices, and  $\overset{\circ}{\nabla}$  denotes the covariant differentiation with respect to the symmetric affine connexion  $\overset{\circ}{\Gamma}_{ji}^k$ .

Similarly, the general connexion in the almost product space with respect to which the almost product structure  $F_j^i$  is covariant constant, has the form:

$$(1.2) \quad \Gamma_{ji}^k = \overset{\circ}{\Gamma}_{ij}^k + \frac{1}{2} (\overset{\circ}{\nabla}_j F_i^a) F_a^k + \frac{1}{2} (\delta_i^a \delta_b^k + F_i^a F_b^k) W_{ja}^b.$$

If we put

$$(1.3) \quad F_a^i F_j^a = \omega \delta_j^i, \quad \omega = +1 \quad \text{or} \quad \omega = -1,$$

and if the operators  $O$  and  $*O$  are defined by ([1]):

$$(1.4) \quad \begin{cases} O_{ir}^{sh} = \frac{1}{2} (\delta_i^s \delta_r^h + \omega F_i^s F_r^h), \\ *O_{ir}^{sh} = \frac{1}{2} (\delta_i^s \delta_r^h - \omega F_i^s F_r^h), \end{cases}$$

the equations (1.1) and (1.2) can be written in the form:

$$(1.5) \quad \Gamma_{ji}^k = \overset{\circ}{\Gamma}_{ji}^k + \frac{1}{2} \omega (\overset{\circ}{\nabla}_j F_i^a) F_a^k + O_{ib}^{ak} W_{ja}^b.$$

In (1.3), (1.4) and (1.5),  $\omega = -1$  if the space is an almost complex space, and  $\omega = +1$  if the space is an almost product space.

The connexion (1.5) being non-symmetric, we can consider four kinds of the covariant derivatives. These covariant derivatives, of a tensor  $a_j^i$  for example, are given by:

$$\overset{1}{\nabla}_k a_j^i = \frac{\partial a_j^i}{\partial x^k} + a_j^p \Gamma_{pk}^i - a_p^i \Gamma_{jk}^p,$$

$$\overset{2}{\nabla}_k a_j^i = \frac{\partial a_j^i}{\partial x^k} + a_j^p \Gamma_{kp}^i - a_p^i \Gamma_{kj}^p,$$

$$\overset{3}{\nabla}_k a_j^i = \frac{\partial a_j^i}{\partial x^k} + a_j^p \Gamma_{pk}^i - a_p^i \Gamma_{kj}^p,$$

$$\overset{4}{\nabla}_k a_j^i = \frac{\partial a_j^i}{\partial x^k} + a_j^p \Gamma_{kp}^i - a_p^i \Gamma_{jk}^p.$$

The connexion (1.5) is the general connexion satisfying the condition  $\overset{2}{\nabla}_k F_j^i = 0$ . In §2 we shall discuss the possibility of finding the  $(F, \overset{1}{F})$ -connexion, i.e. the general affine connexion  $\Gamma_{ji}^k$  such that

$$(1.6) \quad \overset{1}{\nabla}_k F_j^i = \overset{2}{\nabla}_k F_j^i = 0,$$

and in §3 the possibility of finding the  $(F, \overset{3}{F})$ -connexion, i.e. the general affine connexion  $\Gamma_{ji}^k$  such that

$$(1.7) \quad \overset{3}{\nabla}_k F_j^i = \overset{4}{\nabla}_k F_j^i = 0.$$

In §4 we shall show that  $(F, \overset{1}{F})$ -connexion is, under some conditions, Rizza's  $\rho_+$ -connexion and  $(F, \overset{3}{F})$ -connexion is, under some conditions, Rizza's  $\rho_-$ -connexion ([2], [3]) and conversely.

**2.  $(F, \overset{2}{F})$ -connexion.** — Putting

$$(2.1) \quad \Gamma_{ji}^k = \overset{\circ}{\Gamma}_{ji}^k + U_{ji}^k,$$

where

$$(2.2) \quad U_{ji}^k = \frac{1}{2} \omega (\overset{\circ}{\nabla}_j F_i^a) F_a^k + O_{ib}^{ak} W_{ja}^b,$$

we have

$$\overset{1}{\nabla}_k F_j^i = \overset{\circ}{\nabla}_k F_j^i + F_j^p U_{pk}^i - F_p^i \Gamma_{jk}^p,$$

$$\overset{2}{\nabla}_k F_j^i = \overset{\circ}{\nabla}_k F_j^i + F_j^p U_{pk}^i - F_p^i \Gamma_{kj}^p.$$

Thus, the condition (1.6) can be written in the form:

$$F_j^p (U_{kp}^i - U_{pk}^i) - F_p^i (U_{kj}^p - U_{jk}^p) = 0,$$

and consequently

$$(U_{kq}^p - U_{qk}^p) * O_{pj}^{iq} = 0.$$

Substituting (2.2) into this equation, we get

$$\left[ \frac{1}{2} \omega (\overset{\circ}{\nabla}_k F_q^a) F_a^p + O_{qb}^{ap} W_{ka}^b - \frac{1}{2} \omega (\overset{\circ}{\nabla}_q F_k^a) F_a^p - O_{kb}^{ap} W_{qa}^b \right] * O_{pj}^{iq} = 0.$$

Taking into account that  $O_{qb}^{ap} * O_{pj}^{iq} = 0$ , the last equation reduces to

$$\left[ \frac{1}{2} \omega (\overset{\circ}{\nabla}_k F_q^a) F_a^p - \frac{1}{2} \omega (\overset{\circ}{\nabla}_q F_k^a) F_a^p - O_{kb}^{ap} W_{qa}^b \right] * O_{pj}^{iq} = 0.$$

Using Yano's lemma ([1], p. 133), we have

$$\frac{1}{2} \omega (\overset{\circ}{\nabla}_k F_q^a) F_a^p - \frac{1}{2} \omega (\overset{\circ}{\nabla}_q F_k^a) F_a^p - O_{kb}^{ap} W_{qa}^b = O_{qb}^{ap} V_{ka}^b.$$

$V_{ka}^b$  being an arbitrary tensor, we can put

$$V_{ka}^b = -W_{ak}^b.$$

Thus we obtain:

$$(\overset{\circ}{\nabla}_k F_q^a - \overset{\circ}{\nabla}_q F_k^a) F_a^p - (W_{qa}^b F_k^a - W_{ak}^b F_q^a) F_b^p = 0.$$

Contraction by  $F_p^i F_j^k$  gives

$$(2.3) \quad *O_{qj}^{ab} W_{ab}^i = \frac{1}{2} \omega (\overset{\circ}{\nabla}_a F_q^i - \overset{\circ}{\nabla}_q F_a^i) F_j^a.$$

The equation (2.3) admits a solution, by virtue of the Yano's lemma ([1], p. 133), if and only if:

$$O_{qj}^{ab} (\overset{\circ}{\nabla}_i F_a^i - \overset{\circ}{\nabla}_a F_i^i) F_b^i = (\overset{\circ}{\nabla}_a F_q^i - \overset{\circ}{\nabla}_q F_a^i) F_j^a - (\overset{\circ}{\nabla}_a F_j^i - \overset{\circ}{\nabla}_j F_a^i) F_q^a = N_{jq}^i = 0,$$

i. e. iff the structure is integrable. Then the general solution of (2.3) is given by

$$(2.4) \quad W_{ja}^b = \frac{1}{2} \omega (\overset{\circ}{\nabla}_i F_j^b - \overset{\circ}{\nabla}_j F_i^b) F_a^i + O_{ja}^{rs} A_{rs}^b,$$

where  $A_{rs}^b$  is an arbitrary tensor.

Substituting (2.4) into (1.5), we have

$$\Gamma_{ji}^k = \overset{\circ}{\Gamma}_{ji}^k + \frac{1}{2} \omega (\overset{\circ}{\nabla}_j F_i^a) F_a^k + \frac{1}{2} \omega O_{ib}^{ak} (\overset{\circ}{\nabla}_i F_j^b - \overset{\circ}{\nabla}_j F_i^b) F_a^t + O_{ib}^{ak} A_{ja}^{rs} A_{rs}^b,$$

or

$$(2.5) \quad \begin{aligned} \Gamma_{ji}^k = \overset{\circ}{\Gamma}_{ji}^k + \frac{1}{4} \omega [(\overset{\circ}{\nabla}_a F_j^k - \overset{\circ}{\nabla}_j F_a^k) F_i^a + (\overset{\circ}{\nabla}_i F_j^a + \overset{\circ}{\nabla}_j F_i^a) F_a^k] \\ + A_{ji}^k + \omega A_{ai}^b F_j^a F_b^k + \omega A_{ja}^b F_i^a F_b^k + \omega A_{ab}^k F_j^a F_i^b. \end{aligned}$$

( $A_{ji}^k$  being arbitrary, we have put  $A_{ji}^k$  instead of  $\frac{1}{4} A_{ji}^k$ ).

Let  $T_{ji}^k$  be the torsion tensor of the connexion

$$(2.6) \quad \overset{\circ}{\Gamma}_{ji}^k + \frac{1}{4} \omega [(\overset{\circ}{\nabla}_a F_j^k - \overset{\circ}{\nabla}_j F_a^k) F_i^a + (\overset{\circ}{\nabla}_i F_j^a + \overset{\circ}{\nabla}_j F_i^a) F_a^k].$$

Then

$$T_{ji}^k = \frac{1}{4} \omega [(\overset{\circ}{\nabla}_a F_j^k - \overset{\circ}{\nabla}_j F_a^k) F_i^a - (\overset{\circ}{\nabla}_a F_i^k - \overset{\circ}{\nabla}_i F_a^k) F_j^a] = \frac{1}{4} \omega N_{ij}^k = 0,$$

i.e. (2.6) is a symmetric connexion. It will be easily verified that (2.6) is an  $F$ -connexion. Thus we can consider, instead of (2.5), the connexion

$$(2.7) \quad \Gamma_{ji}^k = \overset{\circ}{\Gamma}_{ji}^k + A_{ji}^k + \omega A_{ai}^b F_j^a F_b^k + \omega A_{ja}^b F_i^a F_b^k + \omega A_{ab}^k F_j^a F_i^b,$$

where  $\overset{\circ}{\Gamma}_{ji}^k$  is a symmetric  $F$ -connexion and  $A_{ji}^k$  is an arbitrary tensor.

Thus we have the theorem:

*In order that in an almost complex space or in an almost product space there exists a  $(F, F)$ -connexion, it is necessary and sufficient that the structure be integrable. Then the general  $(F, F)$ -connexion has the form (2.7).*

3.  $(F, F)$ -connexion. — An arbitrary affine connexion may be written as

$$(3.1) \quad \Gamma_{ji}^k = \overset{\circ}{\Gamma}_{ji}^k + A_{ji}^k,$$

where  $\overset{\circ}{\Gamma}_{ji}^k$  is a symmetric affine connexion, and  $A_{ji}^k$  is an arbitrary tensor. Then

$$\overset{3}{\nabla}_k F_j^i = \overset{\circ}{\nabla}_k F_j^i + F_j^p A_{pk}^i - F_p^i A_{kj}^p,$$

and the condition  $\overset{3}{\nabla}_k F_j^i = 0$  is equivalent to

$$(3.2) \quad F_j^p A_{pk}^i - F_p^i A_{kj}^p + \overset{\circ}{\nabla}_k F_j^i = 0.$$

Transvecting (3.2) with  $F_j^j$  and with  $F_i^i$ , we find respectively

$$(3.3) \quad \omega A_{ka}^b F_j^a F_b^i = A_{jk}^i + \omega (\overset{\circ}{\nabla}_k F_a^i) F_j^a,$$

$$(3.4) \quad \omega A_{ak}^b F_j^a F_b^i = A_{kj}^i - \omega (\overset{\circ}{\nabla}_k F_j^a) F_a^i.$$

Multiplying (3.3) by  $F_r^j F_s^k$ , we obtain

$$A_{kr}^b F_b^i F_s^k = A_{jk}^i F_r^j F_s^k + (\overset{\circ}{\nabla}_a F_r^i) F_s^a.$$

Substituting (3.4) into this equation, we have

$$(3.5) \quad \omega A_{ba}^i F_k^b F_j^a = A_{kj}^i - \omega (\overset{\circ}{\nabla}_k F_j^a) F_a^i - \omega (\overset{\circ}{\nabla}_a F_k^i) F_j^a,$$

and consequently

$$\frac{1}{2} A_{ba}^i * O_{kj}^{ba} = \omega [(\overset{\circ}{\nabla}_k F_j^a) F_a^i + (\overset{\circ}{\nabla}_a F_k^i) F_j^a].$$

This equation admits, by the mentioned lemma ([1], p. 133), the solution if and only if

$$[(\overset{\circ}{\nabla}_k F_j^a) F_a^i + (\overset{\circ}{\nabla}_a F_k^i) F_j^a] O_{rs}^{kj} = 0,$$

i. e. if and only if

$$(3.6) \quad (\overset{\circ}{\nabla}_a F_r^i - \overset{\circ}{\nabla}_r F_a^i) F_s^a - (\overset{\circ}{\nabla}_a F_s^i - \overset{\circ}{\nabla}_s F_a^i) F_r^a = N_{sr}^i = 0.$$

This condition shows that the structure  $F_j^i$  must be integrable.

Combining (3.3), (3.4) and (3.5), we get

$$\begin{aligned} 4 A_{kj}^i - 2 \omega (\overset{\circ}{\nabla}_k F_j^a) F_a^i + \omega (\overset{\circ}{\nabla}_j F_a^i) F_k^a - \omega (\overset{\circ}{\nabla}_a F_k^i) F_j^a = \\ = A_{kj}^i + \omega A_{ja}^b F_k^a F_b^i + \omega A_{ak}^b F_j^a F_b^i + \omega A_{ba}^i F_k^b F_j^a, \end{aligned}$$

i. e.

$$\begin{aligned} A_{kj}^i = \frac{1}{4} (A_{kj}^i + \omega A_{ja}^b F_k^a F_b^i + \omega A_{bk}^a F_j^b F_a^i + \omega A_{ba}^i F_k^b F_j^a) + \\ + \frac{\omega}{2} (\overset{\circ}{\nabla}_k F_j^a) F_a^i - \frac{\omega}{4} (\overset{\circ}{\nabla}_j F_a^i) F_k^a + \frac{\omega}{4} (\overset{\circ}{\nabla}_a F_k^i) F_j^a. \end{aligned}$$

Substituting this into (3.1), we find

$$\begin{aligned} \Gamma_{kj}^i = \overset{\circ}{\Gamma}_{kj}^i + \frac{\omega}{2} (\overset{\circ}{\nabla}_k F_j^a) F_a^i - \frac{\omega}{4} (\overset{\circ}{\nabla}_j F_a^i) F_k^a + \frac{\omega}{4} (\overset{\circ}{\nabla}_a F_k^i) F_j^a + \\ + A_{kj}^i + \omega A_{ja}^b F_k^a F_b^i + \omega A_{bk}^a F_j^b F_a^i + \omega A_{ba}^i F_k^b F_j^a, \end{aligned}$$

where  $\overset{\circ}{\Gamma}_{kj}^i$  is a symmetric connexion satisfying (3.6) and where we have put,  $A_{ji}^k$  being arbitrary,  $A_{ji}^k$  instead of  $\frac{1}{4} A_{ji}^k$ .

Let us consider the connexion

$$(3.8) \quad \overset{\circ}{\Gamma}_{kj}^i + \frac{\omega}{2} (\overset{\circ}{\nabla}_k F_j^a) F_a^i - \frac{\omega}{4} (\overset{\circ}{\nabla}_j F_a^i) F_k^a + \frac{\omega}{4} (\overset{\circ}{\nabla}_a F_k^i) F_j^a.$$

It is an  $F$ -connexion, and its torsion tensor

$$\frac{\omega}{4} [(\overset{\circ}{\nabla}_a F_k^i - \overset{\circ}{\nabla}_k F_a^i) F_j^a - (\overset{\circ}{\nabla}_a F_j^i - \overset{\circ}{\nabla}_j F_a^i) F_k^a] = \frac{\omega}{4} N_{jk}^i$$

vanishes. This means that we can consider, instead of (3.7), the connexion

$$(3.9) \quad \Gamma_{kj}^i = \overset{\circ}{\Gamma}_{kj}^i + A_{kj}^i + \omega A_{ja}^b F_k^a F_b^i + \omega A_{bk}^a F_j^b F_a^i + \omega A_{ba}^i F_k^b F_j^a,$$

where  $\overset{\circ}{\Gamma}_{kj}^i$  is a symmetric  $F$ -connexion.

In exactly the same way we can prove that (3.9) is the general connexion satisfying the condition  $\overset{4}{\nabla}_k F_j^i = 0$  too. Thus we have the theorem:

In order that in an almost complex space or in an almost product space there exists a  $(F, F)$ -connexion, it is necessary and sufficient that the structure be integrable. Then the general  $(F, F)$ -connexion has the form (3.9).

4. Rizza's  $\rho_+$ - and  $\rho_-$ -connexions. — G. B. Rizza ([2]) defined  $\rho_+$ - and  $\rho_-$ -connexion in an almost complex space in the following manner.

We can consider, in the tangent space at each point of the manifold with an almost complex structure  $F_j^i$ , the transformations:

$$(4.1) \quad J_\varphi : J_\varphi(U^i) = U^i \cos \varphi + F_a^i U^a \sin \varphi \quad (0 \leq \varphi \leq 2\pi).$$

The torsion of a non-symmetric connexion is the skew-symmetric tensor. Conversely, every skew-symmetric tensor  $S_{ij}^k$  can be considered as the torsion tensor of a non-symmetric connexion. Let us put

$$(4.2) \quad \Omega^k(U, V) = -2 S_{ij}^k U^i V^j,$$

where  $U$  and  $V$  are linearly independent vectors. Applying to  $U$  and  $V$  the transformation  $J_\varphi$ , we may construct the corresponding vector (4.2):

$$\Omega^k(J_\varphi U, J_\varphi V) = -2 S_{ij}^k J_\varphi(U^i) J_\varphi(V^j).$$

On the other hand, we can apply the transformation  $J_\varphi$  to the vector  $\Omega^k(U, V)$ . G. B. Rizza discusses ([2], [3]) the possibility of finding a skew-symmetric tensor  $S_{ij}^k$  such that

$$(4.3) \quad \Omega^k(J_\varphi U, J_\varphi V) = J_\psi \Omega^k(U, V)$$

for arbitrary vectors  $U, V$  and where  $\psi$  is a function of  $\varphi$ . He showed that

$$\text{either } \psi = +2\varphi, \text{ or } \psi = -2\varphi, \text{ or } \psi = 0.$$

The connexion whose torsion tensor satisfies (4.3) such that  $\psi = +2\varphi$  is called an  $\rho_+$ -connexion. The general  $\rho_+$ -connexion has the form ([3]):

$$(4.4) \quad \Gamma_{ij}^k = \overset{\circ}{\Gamma}_{ij}^k + A_{ij}^k + \omega A_{aj}^b F_i^a F_b^k + \omega A_{ia}^b F_j^a F_b^k + \omega A_{ab}^k F_i^a F_j^b,$$

where  $\omega = -1$ ,  $\overset{\circ}{\Gamma}_{ij}^k$  is an arbitrary symmetric connexion and  $A_{ij}^k$  is an arbitrary skew-symmetric tensor.

The connexion whose torsion tensor satisfies (4.3) such that  $\psi = -2\varphi$  is called an  $\rho_-$ -connexion. The general  $\rho_-$ -connexion has the form ([3]):

$$(4.5) \quad \Gamma_{ij}^k = \overset{\circ}{\Gamma}_{ij}^k + A_{ij}^k + \omega A_{ai}^b F_j^a F_b^k + \omega A_{ja}^b F_i^a F_b^k + \omega A_{ab}^k F_i^a F_j^b,$$

where  $\omega$ ,  $\overset{\circ}{\Gamma}_{ij}^k$  and  $A_{ij}^k$  have the same meaning as before.

To define  $\rho_+$ - and  $\rho_-$ -connexion in an  $n$ -dimensional almost product space, we first remember that the structure tensor  $F_j^i$  of a such space defines two complementary distributions in the  $n$ -dimensional tangent plane at each point of the space. Let us suppose that the vector  $U$  does not belong to either of these distributions. Then  $U^i$  and  $F_a^i U^a$  are linearly independent vectors and we can consider, instead of (4.1), the transformations:

$$(4.6) \quad J_\varphi : J_\varphi(U^i) = U^i \operatorname{ch} \varphi + F_a^i U^a \operatorname{sh} \varphi.$$

Supposing that the vector (4.2) does not belong to either of distributions, we may consider the vector  $J_\psi \Omega^k(U, V)$  and then also the condition (4.3), where  $U$  and  $V$  are linearly independent vectors.

The connexion in an almost product space whose torsion tensor satisfies (4.3) for arbitrary vectors  $U, V$  (where  $J_\varphi$  is the transformation (4.6)), such that  $\psi = +2\varphi$  is called an  $\rho_+$ -connexion, and that whose torsion tensor satisfies (4.3) such that  $\psi = -2\varphi$  is called an  $\rho_-$ -connexion. In exactly the same way as in [2] and [3], we obtain that the general  $\rho_+$ -connexion in an almost product space has the form (4.4) (where  $\omega = +1$ ), and the general  $\rho_-$ -connexion has the form (4.5) ( $\omega = +1$ ).

Comparing (4.4) and (2.7), (4.5) and (3.9), we see that:

$\overset{1}{(F, F)}$ -connexion (2.7) in a complex space (in a product space) where arbitrary tensor  $A_{jk}^i$  is skew-symmetric is an  $\rho_+$ -connexion. Conversely, an  $\rho_+$ -connexion (4.4) where  $\overset{0}{\Gamma}_{ij}^k$  is a symmetric  $F$ -connexion, is an  $\overset{1}{(F, F)}$ -connexion.

$\overset{3}{(F, F)}$ -connexion (3.9) in a complex (in a product) space, where  $A_{jk}^i$  is an arbitrary skew-symmetric tensor, is an  $\rho_-$ -connexion. Conversely, an  $\rho_-$ -connexion (4, 5), where  $\overset{0}{\Gamma}_{jk}^i$  is a symmetric  $F$ -connexion, is an  $\overset{3}{(F, F)}$ -connexion.

The  $\rho_+$ - and  $\rho_-$ -connexions in an almost product space cannot be determined if the vector (4.2) belongs to one of the distributions. Thus, we have yet to determine, in the case  $\omega = +1$ , the condition satisfied by torsion tensor  $S_{jk}^i$ , if the vector (4.2) belongs to one of the distributions.

If the vector  $W$  belongs to one of the distributions then either  $W^i = F_a^i W^a$  or  $W^i = -F_a^i W^a$ . This means, in the case of the vector  $S_{jk}^i U^j V^k$ , that

$$S_{jk}^i U^j V^k = \pm F_a^i S_{jk}^a U^j V^k$$

for every pair  $U, V$ . Consequently

$$S_{jk}^i = \pm F_a^i S_{jk}^a.$$

This is the required condition.

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