PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 22 (36), 1977, pp. 211—214

A NOTE ON ORTHOGONAL EXPANSIONS IN MULTIDIMENSIONAL CASE

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(Communicated April 18, 1977)

Let $L^2(I_1 x \dots x I_q)$ be the space of all complex valued locally integrable functions defined on the interval $I_1 x \dots x I_q \subset R^q$, where $I_i=(a_i, b_i)$, $i=1, \dots q$ (I_i may be also the whole R), such that

$$\|f\|=\Big[\int_{I_1 x \dots x I_q} |f(x)|^2 dx\Big]^{\frac{1}{2}}<\infty.$$

In this paper we construct a set of functions in the space $L_2(I_1 x \dots x I_q)$ using complete orthonormal sets of functions from the spaces $L_2(I_i)$, $i=1, \dots q$. Obtained results will be of importance for the sequential theory of some sub-spaces of distributions. Namely, our intention will be in the following papers to make elementary some parts of the theory of distributions analogous to the elementary sequential theory of tempered and periodic distributions [2].

In order to stay in the domain of elementary theory we shall use here only the result from [1].

Let $\{\psi_n^i\}$ be complete orthonormal sets in the spaces $L^2(I_i)$, $i=1, \dots q$. If $x=(\xi_1, \dots, \xi_q) \in I_1 x \dots x I_q$ and $n=(\nu_1, \dots, \nu_q) \in P^q$ we define

$$\psi_n(x)=\psi_{\nu_1}^1(\xi_1) \dots \psi_{\nu_q}^q(\xi_q), \tag{1}$$

where P^q is the set of all non-negative integer points of R^q.
It is easy to verify that

$$\int_{I_1 x \dots x I_q} \psi_m \overline{\psi}_n=0 \text{ for } m\neq n$$

and

$$\int_{I_1 x \dots x I_q} |\psi_n|^2=1 \quad \text{hold,}$$

i.e. the functions ψ_n are orthonormal in $L^2(I_1 x \dots x I_q)$.

E x a m p l e s.

1. Examined cases ([1], [2], [5], [6]):
Hermite functions

$$h_n(x)=(2\pi)^{-\frac{q}{4}}\frac{1}{\sqrt{n!}}e^{-\frac{x^2}{4}}H_{\nu_1}(\xi_1)\ldots H_{\nu_q}(\xi_q)$$

for $x=(\xi_1, \ldots, \xi_q)\in R^q$ and $n=(\nu_1, \ldots, \nu_q)\in P^q$, where $n!=\nu_1!\ldots\nu_q!$, $x^2=\xi_1^2+\cdots+\xi_q^2$ and

$$H_{\nu_i}(\xi_i)=(-1)^{\nu_i}e^{\frac{\xi_i^2}{2}}\left(e^{-\frac{\xi_i^2}{2}}\right)^{(\nu_i)} \quad \text{for } i=1, \ldots, q.$$

Fourier functions

$$(2\pi)^{-\frac{q}{2}}e^{inx}$$

for $x=(\xi_1, \ldots, \xi_q)\in(-\pi. \pi)^q$ and $n=(\nu_1, \ldots, \nu_q)\in B^q$,

where $nx=\sum_{i=1}^{q}\nu_i\xi_i$ and B^q is the set of all integer points of R^q.

2. We can take instead ψ_n in (1) also other orthonormal functions (for ex. [3]) Laguerre's, Jacobi's (Legendre's, Tchebishev's, Gegenbauer's), Bessel's.

3. The functions in (1) need not be of the same type. So we can take for example

$$\psi_n(x)=h_{\nu_1}(\xi_1)(2\pi)^{-\frac{1}{2}}e^{i\nu_2\xi_2}$$

for $x=(\xi_1, \xi_2)\in(-\infty, +\infty)x(-\pi, \pi)$ and $n=(\nu_1, \nu_2)\in PxB$.

Let $\{\varphi_n\}$ be an orthonormal set in $L^2(I_1x\ldots xI_q)$.

We consider series of the form

$$\sum_{n\in P^q}c_n\varphi_n,$$

where the c_n are complex numbers. We can give analogous definition to the definition of unconditionally convergence in square mean for series with Hermite functions (see [2]).

D e f i n i t i o n. *A series $\sum_{n\in P^q}c_n\varphi_n$ converges uncoditionally in square mean to a function $f\in L^2(I_1x\ldots xI_q)$ iff, given any number $\varepsilon>0$, there is a finite set $N_0\subset P^q$ such that*

$$\|f-\sum_{n\in M}c_n\varphi_n\|<\varepsilon$$

for every finite set M such that $N_0\subset M\subset P^q$.

We can formulate the generalization of one part of the theorem 4.8.2 from [2].

Theorem A. *If the series* $\sum_{n\in P^q} c_n\varphi_n$ *converges unconditionally in square mean to* $f\in L^2(I_1 x \dots x I_q)$ *then* $c_n = \int_{I_q x \dots x I_q} f\bar{\varphi}_n$.

The proof is the same as the proof of theorem 4.8.2 from [2], because we use only the orthonormality of $\{\varphi_n\}$.

Now, we formulate the main result.

Main Theorem. *A set of functions* $\{\psi_n\}$ *for* $n\in P^q$ *defined by* (1) *is complete in the space* $L^2(I_1 x \dots x I_q)$.

First, we prove

Lemma. *An orthonormal set* $\{\varphi_k\}$ *for* $k=(k_1, \dots, k_q)\in P^q$ *of functions is complete in the space* $L^2(I_1 x \dots x I_q)$ *iff,*

$$\int_{I_1 x \dots x I_q} h_n^2(x)\,dx = \sum_{k\in P^q}\left|\int_{I_1 x \dots x I_q} h_n(x)\overline{\varphi_k(x)}\,dx\right|^2 \tag{2}$$

holds for all $n\in P^q$.

Proof of the Lemma. That the condition (2) is necessary is obvious.

Assume now, conversely, that (2) holds. It will be of importance the fact, proved in [2], that the set $\{h_n\}$ of Hermite functions is complete in $L^2(R^q)$. Then it is easy to see, that $\{h_n\}$ is also complete in $L^2(I_1 x \dots x I_q)$. Hence, if $f\in L^2(I_1 x \dots x I_q)$, to a given $\varepsilon>0$ there corresponds a finite set $N_0\subset P^q$ such that every finite set M, $N_0\subset M\subset P^q$, there exist complex numbers c_n $(n\in M)$ such that

$$\left\|f-\sum_{n\in M} c_n h_n\right\|<\frac{\varepsilon}{2} \tag{3}$$

holds. We choose an arbitrary finite set $M\supset N_0$. By (2) for every $n\in M$ there exists a finite set $M_n\subset P^q$ such that

$$\left\|c_n h_n - c_n\sum_{k\in M_n}\left(\int_{I_1 x \dots x I_q} h_n(x)\overline{\varphi_k(x)}\,dx\right)\varphi_k\right\|<\frac{\varepsilon}{2\overline{\overline{M}}} \tag{4}$$

holds, where $\overline{\overline{M}}$ is the cardinal number of the set M.

By (3) and (4) we have that

$$\left\|f-\sum_{n\in M} c_n\sum_{k\in M_n}\left(\int_{I_1 x \dots x I_q} h_n\bar{\varphi}_k\right)\varphi_k\right\|\leqslant$$

$$\leqslant\left\|f-\sum_{n\in M} c_n h_n\right\|+\sum_{n\in M}\left\|c_n h_n - c_n\sum_{k\in M_n}\left(\int h_n\bar{\varphi}_k\right)\varphi_k\right\|<\varepsilon.$$

It means that $\{\varphi_n\}$ is closed and hence complete.

Proof of the main theorem. $\{\psi^i_{k_i}\}$, $i=1, \dots, q$, are complete orthonormal sets in the corresponding spaces $L^2(I_i)$ and so by the preceding lemma we have

$$\int_{I_i} h^2_{\nu_i}(\xi_i)\,d\xi_i = \sum_{k_i=0}^{\infty}\left|\int_{I_i} h_{\nu_i}(\xi_i)\,\psi^i_{k_i}(\xi_i)\,d\xi_i\right|^2$$

for every k_i, $\nu_i \in P$, $i=1, \ldots, q$. If we multiply these q equalities one with another we obtain (2) in q-dimensional case. Then from the lemma follows the completeness of $\{\psi_n\}$.

By the main theorem we can generalize also the other part of the theorem 4.8.2 of [2].

Theorem B. *Let* $\{\psi_n\}$ *be the set of functions defined by* (1). *If*

$$f \in L^2(I_1 x \ldots x I_q) \text{ and } c_n = \int_{I_1 x \ldots x I_q} f \bar{\psi}_n,$$

then the series $\sum_{n \in P^q} c_n \psi_n$ *converges unconditionally in square mean to* f.

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