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CERTAIN INTEGRALS INVOLVING THE H-FUNCTION OF SEVERAL VARIABLES

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Summary

In the course of an attempt to unify and extend certain results due to T. M. MacRobert [2], K. C. Sharma [3], H. M. Srivastava and J. P. Singhal [4], and R. Y. Denis [1], an infinite integral is evaluated in terms of the H-function of several variables, which was defined and studied in the recent papers [5] and [6]. A further multiple-integral generalization of this result is also given.

1. Introduction

Following the notation explained fairly fully in our earlier paper [6], let

$$H\begin{matrix} 0,\ \lambda:(\mu',\ \nu');\ \dots;\ (\mu^{(n)},\ \nu^{(n)}) \\ \\ A,\ C:[B',\ D'];\ \dots;\ [B^{(n)},\ D^{(n)}] \end{matrix}\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \tag{1.1}$$

denote the H-function of n complex variables $z_1, \dots, z_n$ (see also [5], p. 271 et seq.). Also, let the associated positive numbers

$$\begin{cases} \theta_j^{(i)},\ j=1, \dots, A; & \varphi_j^{(i)},\ \ j=1, \dots,\ B^{(i)} \\ \psi_j^{(i)},\ j=1, \dots, C; & \delta_j^{(i)},\ \ j=1, \dots,\ D^{(i)};\quad 1 \leqq i \leqq n; \end{cases} \tag{1.2}$$

be constrained by the inequalities

$$\begin{aligned} \Lambda_i \equiv \sum_{j=1}^{\lambda} \theta_j^{(i)} - \sum_{j=\lambda+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}} \varphi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}} \varphi_j^{(i)} \\ - \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0, \end{aligned} \tag{1.3}$$

$$\lambda_i \equiv \sum_{j=1}^{A} \theta_j^{(i)} + \sum_{j=1}^{B^{(i)}} \varphi_j^{(i)} - \sum_{j=1}^{C} \psi_j^{(i)} - \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} < 0, \qquad \forall i \in \{1, \dots, n\}. \tag{1.4}$$

Then it is known that the multiple Mellin-Barnes contour integral [5, p. 271, Eq. (4.1)] defining the function (1.1) would converge absolutely when

$$(1.5) \qquad |\arg(z_i)| < \frac{1}{2}\Lambda_i \pi, \qquad i=1, \ldots, n,$$

it being understood that the points $z_i = 0$, $i = 1, \ldots, n$, are excluded, and that (cf. [6], p. 122, Eq. (1.16))

$$(1.6) \qquad H^{0,\, \lambda:(\mu',\, \nu');\, \ldots;\, (\mu^{(n)},\, \nu^{(n)})}_{A,\, C:[B',\, D'];\, \ldots;\, [B^{(n)},\, D^{(n)}]} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

$$= \begin{cases} O(|z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n}), & \max\{|z_1|, \ldots, |z_n|\} \to 0, \\ O(|z_1|^{-\beta_1} \cdots |z_n|^{-\beta_n}), & \lambda \equiv 0,\ \min\{|z_1|, \ldots, |z_n|\} \to \infty, \end{cases}$$

where, with $i = 1, \ldots, n$,

$$(1.7) \qquad \begin{cases} \alpha_i = d_j^{(i)}/\delta_j^{(i)}, & j = 1, \ldots, \mu^{(i)}, \\ \beta_i = [1 - b_j^{(i)}]/\varphi_j^{(i)}, & j = 1, \ldots, \nu^{(i)}. \end{cases}$$

The main result of the present paper is the following infinite integral:

$$(1.8) \qquad \int_0^\infty x^{\beta-1}(x+y)^{-\alpha-\beta} H^{0,\, 0:(\mu',\, \nu');\, \ldots;\, (\mu^{(n)},\, \nu^{(n)})}_{A,\, C:[B',\, D'];\, \ldots;\, [B^{(n)},\, D^{(n)}]} \begin{pmatrix} z_1 (x+y)^{-\rho_1} x^{\rho_1 - \sigma_1} \\ \vdots \\ z_n (x+y)^{-\rho_n} x^{\rho_n - \sigma_n} \end{pmatrix} dx$$

$$= y^{-\alpha} H^{0,\, 2:(\mu',\, \nu');\, \ldots;\, (\mu^{(n)},\, \nu^{(n)})}_{A+2,\, C+1:[B',\, D'];\, \ldots;\, [B^{(n)},\, D^{(n)}]}$$

$$\left(\begin{matrix} [1-\alpha : \sigma_1, \ldots, \sigma_n],\ [1-\beta : \rho_1 - \sigma_1, \ldots, \rho_n - \sigma_n],\ [(a) : \theta', \ldots, \theta^{(n)}] : \\ [1-\alpha-\beta : \rho_1, \ldots, \rho_n],\ [(c) : \psi', \ldots, \psi^{(n)}] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \varphi'];\, \ldots;\, [(b^{(n)}) : \varphi^{(n)}]; \\ [(d') : \delta'];\, \ldots;\, [(d^{(n)}) : \delta^{(n)}]; \end{matrix} \ z_1/y^{\sigma_1}, \ldots, z_n/y^{\sigma_n} \right),$$

provided that (1.3), (1.4) and (1.5) hold, and

$$(1.9) \qquad \mathrm{Re}\left(\alpha + \sum_{i=1}^n \sigma_i \beta_i\right) > 0, \quad \mathrm{Re}\left\{\beta + \sum_{i=1}^n (\rho_i - \sigma_i)\alpha_i\right\} > 0,$$

where $\rho_i > \sigma_i > 0$, $i = 1, \ldots, n$, and α_i and β_i are given by (1.7).

We also give an analogue of our integral formula (1.8) in terms of multiple integrals.

2. Evaluation of (1.8)

Our derivation of the integral formula (1.8) makes use of the following well-known integral:

$$\int_0^\infty x^{\alpha-1}(x+y)^{-\beta}\,dx=\frac{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}{\Gamma(\beta)}\,y^{\alpha-\beta},$$

which holds when $\mathrm{Re}(\beta)>\mathrm{Re}(\alpha)>0$. Indeed, we first replace the multiple H-function in the integrand of (1.8) by its Mellin- Barnes contour integral [5, p. 271, Eg. (4.1)], and change the order of integration, which is permissible under the conditions stated with (1.8). We then evaluate the innermost x-integral by applying (2.1), and interpret the resulting multiple contour integral as an H-function of several variables. The final result (1.8), together with the aforementioned conditions of its convergence, will follow from the asymptotic expansions given by (1.6) above.

3. Extensions and particular cases

By setting each of the positive coefficients in (1.2) equal to 1, our integral formula (1.8) can easily be rewritten in terms of the G-function of several variables. Thus, for $n=1$, our result (1.8) would provide a generalization of the integral formulas involving E and G functions, given earlier by MacRobert [2] and Sharma [3], respectively. On the other hand, for $n=2$, it would yield (as special cases) the integrals involving the G-function of two variables evaluated by Srivastava and Singhal [4], and subsequently also by Denis [1]. We choose to omit the details of these and several other interesting specializations of our result (1.8).

Next we consider the special case of our integral (1.8) when $A=C=0$. In this case the multiple H-function on the left-hand side would reduce to the product of n H-functions of different arguments, and our result would evidently yield a formula for the corresponding infinite integral involving the product of several H-functions.

Finally, we remark that the method of derivation of the integral formula (1.8) can be applied to obtain the following multiple-integral analogue:

$$(3.1)\qquad \int_0^\infty\cdots\int_0^\infty \prod_{j=1}^{n}\left\{x_j^{u_j-1}(x_j+y_j)^{-u_j-v_j}\right\}$$

$$\cdot H\,{}^{0,\,0:(\mu',\,\nu');\,\ldots;\,(\mu^{(n)},\,\nu^{(n)})}_{A,\,C:[B'\,D'];\,\ldots;\,[B^{(n)},\,D^{(n)}]}\begin{pmatrix} z_1(x_1+y_1)^{-\rho_1}x_1^{\rho_1-\sigma_1}\\ \vdots\\ z_n(x_n+y_n)^{-\rho_n}x_n^{\rho_n-\sigma_n}\end{pmatrix}$$

$$\cdot\,dx_1\cdots dx_n$$

$$= y^{-u_1-\cdots-u_n} H \begin{matrix} 0,\, 2n : (\mu', \nu'); \ldots; (\mu^{(n)}, \nu^{(n)}) \\ A+2n,\, C+n : [B', D']; \ldots; [B^{(n)}, D^{(n)}] \end{matrix}$$

$$\left(\begin{matrix} [1-u_1 : \sigma_1, \ldots, \sigma_n],\ [1-v_1 : \rho_1-\sigma_1, \ldots, \rho_n-\sigma_n], \ldots, [1-u_n : \sigma_1, \ldots, \sigma_n], \\ [1-v_n : \rho_1-\sigma_1, \ldots, \rho_n-\sigma_n], \\ [1-u_1-v_1 : \rho_1, \ldots, \rho_n], \ldots, [1-u_n-v_n : \rho_1, \ldots, \rho_n], \\ [(a) : \theta', \ldots, \theta^{(n)}] : [(b') : \varphi']; \ldots; [(b^{(n)}) : \varphi^{(n)}]; \\ [(c) : \psi', \ldots, \psi^{(n)}] : [(d') : \delta']; \ldots; [(d^{(n)}) : \delta^{(n)}]; \end{matrix} \ \frac{z_1}{y_1^{\sigma_1}}, \ldots, \frac{z_n}{y_n^{\sigma_n}} \right)$$

where $\rho_i > \sigma_i > 0$, $i = 1, \ldots, n$, and, for convergence,

$$\text{(3.2)} \qquad \operatorname{Re}\left(u_j + \sum_{i=1}^{n} \sigma_i \beta_i\right) > 0,$$

$$\text{(3.3)} \qquad \operatorname{Re}\left\{v_j + \sum_{i=1}^{n} (\rho_i - \sigma_i)\alpha_i\right\} > 0, \quad j = 1, \ldots, n.$$

α_i and β_i being given by (1.7).

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