

A SINGULAR CONVOLUTION EQUATION IN THE SPACE  
 OF DISTRIBUTIONS. II

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(Received June 15, 1977)

In theory of the distributional convolution process the equation  $R * T = S$  is fundamental ( $R, S$  given distributions,  $T$  unknown). Its solution may be given if there exists an associate convolution algebra. For  $R = vp \frac{1}{t}$ ,  $S \in \mathcal{O}'_\alpha$  the solution is established in [1] applying the Plemelj distributional formulas and the distributional analytic continuation.

The purpose of this paper is to complete the solution of the equation

$$(1) \quad a(t)T + \frac{b(t)}{\pi i} \left( T * vp \frac{1}{t} \right) = S$$

studied in [1], and thereafter to solve *an equation adjoint* to (1).

1. The equation (1) is solved under assumption that the index

$$\lambda = \frac{1}{2\pi i} \left\{ \log \frac{a(t) + b(t)}{a(t) - b(t)} \right\}_{\bar{R}}$$

is nonnegative. Now it will be shown that for  $\lambda < 0$  a solution exists only under certain conditions. It is known ([1]) that in the case  $\lambda < 0$  the solution of the Hilbert boundary problem has the form

$$\hat{T}(z) = X(z) \hat{S}(z),$$

where

$$\hat{S}(z) = \frac{1}{2\pi i} \left\langle \frac{S}{X^+(\tau)[a(\tau) - b(\tau)]}, \frac{1}{\tau - z} \right\rangle \quad (z \in \Delta^\pm).$$

But the canonical function

$$X^-(z) = \left( \frac{z-i}{z+i} \right)^{-\lambda} \exp \Gamma_\lambda^-(z) \quad (z \in \Delta^-)$$

has a pole of order  $(-\lambda)$  at point  $z = -i$ . For the equation (1) to be soluble ( $\lambda < 0$ ) the function  $\hat{T}(z)$  must be holomorphic. Clearly this will be accomplished if the function

$$\frac{\hat{S}(z)}{(z+i)^{-\lambda}}$$

is holomorphic. Let us develop  $\hat{S}(z)$  in the Taylor series about the point  $z = -i$ . The radius of convergence is equal to the shortest distance from the point  $z = -i$  to  $\text{Supp } S$ . Since the coefficients of the series are defined by the derivatives

$$S^{(n)}(z) = \frac{n!}{2\pi i} \left\langle \frac{S}{X^+(\tau)[a(\tau)-b(\tau)]}, \frac{1}{(\tau-z)^{n+1}} \right\rangle \quad (z \in \Delta^\pm),$$

it follows that the function  $\hat{T}(z)$  is holomorphic at the point  $z = -i$  if and only if the distribution  $S$  satisfies the  $(-\lambda)$  conditions

$$(2) \quad \left\langle \frac{S}{X^+(\tau)[a(\tau)-b(\tau)]}, \frac{1}{(\tau+i)^k} \right\rangle = 0 \quad (k = 1, 2, \dots, (-\lambda)).$$

Let

$$a^*(t) = \frac{a(t)}{a^2(t) - b^2(t)},$$

$$b^*(t) = \frac{b(t)}{a^2(t) - b^2(t)},$$

$$Z(t) = X^+(t)[a(t) - b(t)] = X^-(t)[a(t) + b(t)].$$

After a simple transformation the solution  $T$  ([1], Relation (19)) may be written as

$$T = a^*(t)S - \frac{Z(t)b^*(t)}{\pi i} \left( \frac{S}{Z(t)} * vp \frac{1}{t} \right) + Z(t)b^*(t) \frac{P_{\lambda-1}(t)}{(t+i)^\lambda}.$$

Consequently, this formula giving the general solution of (1) for  $\lambda \geq 0$  also gives the solution for  $\lambda < 0$  if one puts  $P_{\lambda-1}(t) \equiv 0$  and assumes that  $S$  satisfies the necessary and sufficient conditions (2).

In particular, for  $S = \delta$  and  $\lambda = 0$ , the solution of (1) has the form

$$T = a^*(0)\delta - \frac{Z(t)b^*(t)}{\pi i Z(0)} vp \frac{1}{t}.$$

**2.** Before formulating the equation adjoint to (1), let us recall an essential definition.

The function  $f: \mathbf{R} \rightarrow \mathbf{C}$  is said to satisfy the Hölder ( $H$ ) condition at infinity for the constant  $\mu > 0$  if  $f(t) - f(\infty) = O\left(\frac{1}{|t|^\mu}\right)$ , where

$$f(+\infty) = \lim_{t \rightarrow +\infty} f(t), \quad f(-\infty) = \lim_{t \rightarrow -\infty} f(t), \quad f(+\infty) = f(-\infty) = f(\infty) \in \mathbf{C}.$$

When it is necessary to specify the constant  $\mu$  the  $H$  condition at infinity will be denoted by  $H(\mu, \infty)$ .

The definition implies the following assertions (which will be used implicitly):

The continuous function  $f: \mathbf{R} \rightarrow \mathbf{C}$  satisfying the  $H$  condition at infinity is bounded on  $\mathbf{R}$ .

If the continuous functions  $f, g: \mathbf{R} \rightarrow \mathbf{C}$  satisfy the  $H(\mu, \infty)$  and  $H(\eta, \infty)$  conditions respectively, then the functions  $f+g, f \cdot g$  satisfy the  $H$  condition for the constant equal to  $\min(\mu, \eta)$ ; if  $f(t) \neq 0$  on  $\overline{\mathbf{R}}$ , then  $\frac{1}{f}$  satisfies the  $H(\mu, \infty)$  condition. Moreover, if

$$\frac{1}{2\pi i} \{\log f(t)\}_{\overline{\mathbf{R}}} = 0,$$

then  $\log f(t)$  also satisfies the same condition (note that the last relation implies the single-valuedness of the logarithm).

**Problem.** Let  $a(t)$  and  $b(t)$  be given complex-valued functions in  $C^\infty(\mathbf{R})$  which satisfy with all their derivatives the  $H(\mu, \infty)$  and  $H(\eta, \infty)$  conditions, respectively. In addition, one supposes that  $a(t) \pm b(t) \neq 0$  on  $\overline{\mathbf{R}}$ . Let  $B$  be a given distribution in  $\mathcal{E}'$ . Find the distribution  $A \in \mathcal{O}'_\alpha$  with  $-1 \leq \alpha < 0$  such that

$$(3) \quad a(t) A - \frac{1}{\pi i} \left( b(t) A * vp \frac{1}{t} \right) = B.$$

**Solution.** First of all let us observe that the functions  $a(t)$  and  $b(t)$  are multipliers of  $\mathcal{O}'_\alpha$ . After a detailed discussion in [1] it can be introduced at once the locally holomorphic function

$$\hat{A}(z) = \frac{1}{2\pi i} \left\langle b(\tau) A, \frac{1}{\tau - z} \right\rangle \quad (z \in \Delta^\pm)$$

vanishing at infinity. In view of the formulas

$$\hat{A}^+ - \hat{A}^- = b(t) A,$$

$$\hat{A}^+ + \hat{A}^- = - \left( b(t) A * vp \frac{1}{t} \right),$$

true in the  $\mathcal{O}'_\alpha$  topology ( $-1 \leq \alpha < 0$ ) it may be shown that the solution of the equation (3) is equivalent to the following boundary problem:

Find a distribution  $A$  and a locally holomorphic function  $\hat{A}(z)$ , vanishing at infinity, subject to the conditions

$$b(t) A = \hat{A}^+ - \hat{A}^-, \quad a(t) A = - \hat{A}^+ - \hat{A}^- + B.$$

Since the logarithm in (6) together with all its derivatives satisfies the  $H$  condition on compacts in  $\mathbb{R}$  and at infinity, the Peierls formulas for each derivative of  $T^y(z)$  hold. This implies that  $X_+(t)$  and  $X_-(t)$  belongs to the subspace of  $C^\infty(\mathbb{R})$  consisting of elements bounded on  $\mathbb{R}$  and different from zero. Therefore these functions are multipliers for  $\mathcal{Q}_0$ , and (8) is well-defined.

$$(8) \quad \frac{X_+(t) - X_-(t)}{A_+} = \frac{a(t) + b(t)}{b(t)B}.$$

the problem (5) becomes

$$\frac{X_-(t)}{X_+(t)} = \frac{a(t) + b(t)}{a(t) - b(t)}$$

According to the relation

$$(7) \quad X(z) = \frac{z}{1 - \sum_{\pm} \nabla_{\pm}}.$$

Since  $T^y(z) = -T^a(z)$ , comparing the present canonical functions with those defined by  $T^a(z)$  one gets

$$(6) \quad T^y(z) = \frac{1}{2\pi i} \int_z^{-\infty} \log \left[ \frac{a(\tau) + b(\tau)}{a(\tau) - b(\tau)} \right] d\tau.$$

be canonical functions of the problem (5), where

$$X_+(z) = \exp T^y_+(z), \quad X_-(z) = \exp T^y_-(z)$$

Suppose  $v = -\chi \leq 0$ . Let

$$v = \frac{1}{2\pi i} \left\{ \log \frac{a(t) + b(t)}{a(t) - b(t)} \right\} = -\chi.$$

Evidently, the coefficient of this last problem is equal to the reciprocal coefficient of the Hilbert problem according to the equation (1). Hence

$$(5) \quad A_+ = \frac{a(t) + b(t)}{a(t) - b(t)} \frac{A_-}{A_+} + \frac{a(t) + b(t)}{b(t)B}.$$

Comparing the right sides one is led to the Hilbert problem

$$(4) \quad A = \frac{a(t) + b(t)}{2A_+} + \frac{B}{A_+}, \quad A = \frac{a(t) - b(t)}{2A_+} + \frac{a(t) - b(t)}{B}.$$

or

$$[a(t) + b(t)]A = -2A_- + B, \quad [a(t) - b(t)]A = -2A_+ + B,$$

Adding and subtracting one obtains the equivalent boundary conditions

Paralleling the proceeding in [1] and using (7) we obtain

$$(9) \quad \hat{A}(z) = \frac{1}{X(z)} \left\{ \frac{1}{2\pi i} \left\langle \frac{X^+(\tau) b(\tau) B}{a(\tau) + b(\tau)}, \frac{1}{\tau - z} \right\rangle + Q_{v-1}(z) \right\} \quad (z \in \Delta^\pm),$$

where  $Q_{v-1}(z)$  is an arbitrary polynomial of degree not greater than  $v-1$  ( $Q_{v-1}(z) \equiv 0$  for  $v=0$ ).

Suppose  $v = -\lambda < 0$  ( $\lambda > 0$ ). In this case the function  $\hat{A}(z)$  is also given by (9) with  $Q_{v-1}(z) \equiv 0$ , if the following necessary and sufficients conditions are satisfied:

$$(10) \quad \left\langle \frac{X^+(\tau) b(\tau) B}{a(\tau) + b(\tau)}, \frac{1}{(\tau + i)^k} k \right\rangle = 0 \quad (k = 1, 2, \dots, (-v)).$$

Finally, let us calculate  $\hat{A}^+$  or  $\hat{A}^-$  from (9) by the Plemelj formulas. In view of the relations

$$\begin{aligned} \frac{B}{a(t) - b(t)} - \frac{b(t) B}{a^2(t) - b^2(t)} &= a^*(t) B, \\ \frac{X^+(t) B}{a(t) + b(t)} &= Z(t) b^*(t), \end{aligned}$$

one obtains for  $v \geq 0$  by either of the formulas (4) the solution of the equation (3):

$$(11) \quad A = a^*(t) B + \frac{1}{\pi i Z(t)} \left( Z(t) b^*(t) B^* v p \frac{1}{t} \right) + \frac{1}{Z(t)} \frac{P_{v-1}(t)}{(t+i)^v}.$$

Here for  $v=0$  it is necessary to put  $P_{v-1}(t) \equiv 0$ . If  $v < 0$  the solution is also given by (11) with  $P_{v-1}(t) \equiv 0$  if and only if the distribution  $B$  satisfies the conditions (10).

#### REFERENCE

- [1] D. Mitrović, *A singular convolution equation in the space of distributions*, Publ. Inst. Math. t. 21 (35) 1977 pp. 151—163.

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