

NEW COMMUTATION FORMULAS IN THE NON-SYMMETRIC AFFINE CONNEXION SPACE

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Summary

In an earlier work [1], using two kinds of covariant derivations, we obtained 10 commutation formulas of Ricci type for alternated covariant derivation of second degree in the non-symmetric affine connexion space L_N . In this way we obtained 3 curvature tensors and 15 magnitudes which we called "curvature pseudotensors" of the space L_N .

In the present work we use two new kinds of covariant derivation and, in this way, we get 10 new Ricci type identities, i.e. new commutation formulas. In this case, besides above mentioned tensors and pseudotensors, there appears a new curvature tensor $(R)_4$.

0. In the non-symmetric affine connexion space L_N we can observe 4 kinds of covariant derivation. For example, for the tensor a_{kl}^{hij} we have

$$(1) \quad \begin{aligned} a_{kl|_m}^{hij} &= a_{kl,m}^{hij} + L_{pm}^h a_{kl}^{pij} + L_{pm}^i a_{kl}^{hpj} + L_{pm}^j a_{kl}^{hip} - \\ &\quad - L_{km}^p a_{pl}^{hij} - L_{lm}^p a_{kp}^{hij}, \end{aligned}$$

$$(2) \quad \begin{aligned} a_{kl|_m}^{hij} &= a_{kl,m}^{hij} + L_{mp}^h a_{kl}^{pij} + L_{mp}^i a_{kl}^{hpj} + L_{mp}^j a_{kl}^{hip} - \\ &\quad - L_{mk}^p a_{pl}^{hij} - L_{ml}^p a_{kp}^{hij}, \end{aligned}$$

$$(3) \quad \begin{aligned} a_{kl|_m}^{hij} &= a_{kl,m}^{hij} + L_{pm}^h a_{kl}^{pij} + L_{pm}^i a_{kl}^{hpj} + L_{pm}^j a_{kl}^{hip} - \\ &\quad - L_{mk}^p a_{pl}^{hij} - L_{ml}^p a_{kp}^{hij}, \end{aligned}$$

$$(4) \quad \begin{aligned} a_{kl|_m}^{hij} &= a_{kl,m}^{hij} + L_{mp}^h a_{kl}^{pij} + L_{mp}^i a_{kl}^{hpj} + L_{mp}^j a_{kl}^{hip} - \\ &\quad - L_{km}^p a_{pl}^{hij} - L_{lm}^p a_{kp}^{hij}, \end{aligned}$$

where the comma (,) signifies partial derivation, for example

$$a_{kl,m}^{hij} = \frac{\partial}{\partial x^m} a_{kl}^{hij}.$$

In the work [1] we obtained 10 commutation formulas, using the covariant derivations of 1st and 2nd kind. In this case we obtained 3 curvature tensors R, R, R and 15 magnitudes A_1, \dots, A_{15} , we call them "curvature pseudotensors".

In the present work we observe Ricci type identities which one obtains by the covariant derivations of 3rd and 4th kind (3), (4).

1. On the base of (3), we obtain

$$\begin{aligned} a_{kl}^{hij}{}_{mn} = & (a_{kl}^{hij}{}_{m})_{|n} = (a_{kl}^{hij}{}_{mn})_{,n} + L_{sn}^h a_{kl}^{sij}{}_{|m} + \\ & + L_{sn}^i a_{kl}^{hsj}{}_{|m} + L_{sn}^j a_{kl}^{his}{}_{|m} - L_{nk}^s a_{sl}^{hij}{}_{|m} - L_{nl}^s a_{ks}^{hij}{}_{|m} - L_{nm}^s a_{kl}^{hij}{}_{|s}, \end{aligned}$$

i.e.

$$\begin{aligned} (5) \quad a_{kl}^{hij}{}_{mn} = & a_{kl}^{hij}{}_{mn} + L_{pm,n}^h a_{kl}^{pij} + L_{pm}^h a_{kl,n}^{pij} + L_{pm,n}^i a_{kl}^{hpj} + \\ & + L_{pm}^i a_{kl,n}^{hpj} + L_{pm,n}^j a_{kl}^{hip} + L_{pm}^j a_{kl,n}^{hip} - L_{mk,n}^p a_{pl}^{hij} - L_{mk}^p a_{pl,n}^{hij} - \\ & - L_{ml,n}^p a_{kp}^{hij} - L_{ml}^p a_{kp,n}^{hij} + L_{sn}^h a_{kl,m}^{sij} + L_{sn}^h L_{pm}^s a_{kl}^{pis} + L_{sn}^h L_{pm}^i a_{kl}^{spj} + \\ & + L_{sn}^h L_{pm}^j a_{kl}^{sip} - L_{sn}^h L_{mk}^p a_{pl}^{sij} - L_{sn}^h L_{ml}^p a_{kp}^{sij} + L_{sn}^i a_{kl,m}^{hsj} + \\ & + L_{sn}^i L_{pm}^h a_{kl}^{psj} + L_{sn}^i L_{pm}^s a_{kl}^{hpj} + L_{sn}^i L_{pm}^j a_{ki}^{hsp} - L_{sn}^i L_{mk}^p a_{pl}^{hsj} - \\ & - L_{sn}^i L_{ml}^p a_{kp}^{hsj} + L_{sn}^j a_{kl,m}^{his} + L_{sn}^j L_{pm}^h a_{kl}^{pis} + L_{sn}^j L_{pm}^i a_{kl}^{hps} + \\ & + L_{sn}^j L_{pm}^s a_{kl}^{hip} - L_{sn}^j L_{mk}^p a_{pl}^{his} - L_{sn}^j L_{ml}^p a_{kp}^{his} - L_{nk}^s a_{sl,m}^{hij} - \\ & - L_{nk}^s L_{pm}^h a_{sl}^{pij} - L_{nk}^s L_{pm}^i a_{sl}^{hpj} - L_{nk}^s L_{pm}^j a_{sl}^{hip} + L_{nk}^s L_{ms}^p a_{pl}^{hij} + \\ & + L_{nk}^s L_{ml}^p a_{kp}^{hij} - L_{nl}^s a_{ks,n}^{hij} - L_{nl}^s L_{pm}^h a_{ks}^{pij} - L_{nl}^s L_{pm}^i a_{ks}^{hpj} - \\ & - L_{nl}^s L_{pm}^j a_{ks}^{hip} + L_{nl}^s L_{mk}^p a_{ps}^{hij} + L_{nl}^s L_{ms}^p a_{kp}^{hij} - L_{nm}^s a_{kl,s}^{hij}, \end{aligned}$$

from where one obtains

$$\begin{aligned} (6) \quad a_{kl}^{hij}{}_{mn} - a_{kl}^{hij}{}_{nm} = & R_{pmn}^h a_{kl}^{pij} + R_{pmn}^i a_{kl}^{hpj} + R_{pmn}^j a_{kl}^{hip} - \\ & - R_{kmm}^p a_{pl}^{hij} - R_{lmn}^p a_{kp}^{hij} + 2 \underset{\swarrow}{L}_{mn}^p a_{kl,p}^{hij}, \end{aligned}$$

where mn designates the antisymmetrisation over m, n , for example

$$(7) \quad \underset{\swarrow}{L}_{mn}^p = \frac{1}{2} (L_{mn}^p - L_{nm}^p)$$

and

$$(8) \quad R_{jmn}^i = L_{jm,n}^i - L_{jn,m}^i + L_{jm}^p L_{pn}^i - L_{jn}^p L_{pm}^i,$$

$$(9) \quad R_{jmn}^i = L_{mj,n}^i - L_{nj,m}^i + L_{mj}^p L_{np}^i - L_{nj}^p L_{mp}^i,$$

are the curvature tensors of 1st and 2nd kind respectively in the space L_N .

Generally, it is

$$(10) \quad a_{t_1 \dots t_v}^{r_1 \dots r_u} - a_{t_1 \dots t_v}^{r_1 \dots r_u} = \sum_{l=1}^u R_{pmn}^{r_l} \binom{p}{r_l} a_{\dots} - \sum_{f=1}^v R_{tfmn}^p \binom{t_f}{p} a_{\dots} + 2 L_{mn}^p a_{t_1 \dots t_v}^{r_1 \dots r_u},$$

where we have put

$$(11a) \quad \binom{p}{r_l} a_{\dots} = a_{t_1 \dots t_y}^{r_1 \dots r_{l-1} p r_{l+1} \dots r_u},$$

$$(11b) \quad \binom{t_f}{p} a_{\dots} = a_{t_1 \dots t_{f-1} p t_{f+1} \dots t_y}^{r_1 \dots r_u}.$$

2. Using (4), we get

$$\begin{aligned} a_{kl}^{hij} &= (a_{kl}^{hij})_{|mn} = (a_{kl}^{hij})_{|n} + L_{ns}^h a_{kl}^{sij} + \dots \\ &- L_{kn}^s a_{sl}^{hij} - L_{ln}^s a_{ks}^{hij} - L_{mn}^s a_{kl}^{hij}, \end{aligned}$$

i.e.

$$\begin{aligned} (12) \quad a_{kl}^{hij} &= a_{kl}^{hij} + L_{mp,n}^h a_{kl}^{pij} + L_{mp}^h a_{kl,n}^{pij} + L_{mp,n}^i a_{kl}^{hpi} + \\ &+ L_{mp}^i a_{kl,n}^{hpi} + L_{mp,n}^j a_{kl}^{hip} + L_{mp}^j a_{kl,n}^{hip} - L_{km,n}^p a_{pl}^{hij} - L_{km}^p a_{pl,n}^{hij} - \\ &- L_{lm,n}^p a_{kp}^{hij} - L_{lm}^p a_{kp,n}^{hij} + L_{ns}^h a_{kl,m}^{sij} + L_{ns}^h L_{mp}^s a_{kl}^{pij} + L_{ns}^h L_{mp}^i a_{kl}^{spj} + \\ &+ L_{ns}^h L_{mp}^j a_{kl}^{sip} - L_{ns}^h L_{km}^p a_{pl}^{sij} - L_{ns}^h L_{lm}^p a_{kp}^{sij} + L_{ns}^i a_{kl,m}^{hsj} + \\ &+ L_{ns}^i L_{mp}^h a_{kl}^{psj} + L_{ns}^i L_{mp}^s a_{k}^{hpj} + L_{ns}^i L_{mp}^j a_{kl}^{hsp} - L_{ns}^i L_{km}^p a_{pl}^{hsj} - \\ &- L_{ns}^i L_{lm}^p a_{kp}^{hsj} + L_{ns}^j a_{kl,m}^{his} + L_{ns}^j L_{mp}^h a_{kl}^{pis} + L_{ns}^j L_{mp}^i a_{kl}^{hps} + \\ &+ L_{ns}^j L_{mp}^s a_{kl}^{hip} - L_{ns}^j L_{km}^p a_{pl}^{his} - L_{ns}^j L_{lm}^p a_{kp}^{his} - L_{kn}^s a_{sl,m}^{hij} - \\ &- L_{kn}^s L_{mp}^h a_{sl}^{pij} - L_{kn}^s L_{mp}^i a_{sl}^{hpi} - L_{kn}^s L_{mp}^j a_{sl}^{hip} + L_{kn}^s L_{sm}^p a_{pl}^{hij} + \\ &+ L_{kn}^s L_{lm}^p a_{sl}^{hij} - L_{ln}^s a_{ks,m}^{hij} - L_{ln}^s L_{mp}^h a_{sl}^{hpi} - L_{ln}^s L_{mp}^i a_{ks}^{hpi} - \\ &- L_{ln}^s L_{mp}^j a_{ks}^{hip} + L_{ln}^s L_{km}^p a_{ps}^{hij} + L_{ln}^s L_{sm}^p a_{kp}^{hij} - L_{mn}^s a_{kl,s}^{hij}, \end{aligned}$$

from where it is

$$(13) \quad a_{kl}^{hij} - a_{kl}^{hij} = R^h_{pmn} a_{kl}^{pij} + \dots - R^p_{kmn} a_{pl}^{hij} - \\ - R^p_{lma} a_{kp}^{hij} - 2 L_{mn}^p a_{kl}^{hij}.$$

Generally:

$$(14) \quad a_{t_1 \dots t_v}^{r_1 \dots r_u} - a_{t_1 \dots t_v}^{r_1 \dots r_u} = \sum_{l=1}^u R^r_{lpmn} \binom{p}{r_l} a_{\dots} - \\ - \sum_{f=1}^v R^p_{tfmn} \binom{t_f}{p} a_{\dots} - 2 L_{mn}^p a_{t_1 \dots t_v}^{r_1 \dots r_u}.$$

3. Combining the 3rd and 4th kind of derivation on the base of (3) and (4), we have

$$a_{kl}^{hij} - a_{kl}^{hij} = (a_{kl}^{hij})_{,n} + L_{ns}^h a_{kl}^{sij} + L_{ns}^i a_{kl}^{hsj} + L_{ns}^j a_{kl}^{his} - \\ - L_{kn}^s a_{sl}^{hij} - L_{ln}^s a_{ks}^{hij} - L_{mn}^s a_{kl}^{hij},$$

and further

$$(15) \quad a_{kl}^{hij} - a_{kl}^{hij} = a_{kl,mn}^{hij} + L_{pm,n}^h a_{kl}^{pij} + L_{pm,n}^h a_{kl}^{pij} + L_{pm,n}^i a_{kl}^{hpj} + \\ + L_{pm}^i a_{kl}^{hpj} + L_{pm,n}^j a_{kl}^{hpj} + L_{pm}^j a_{kl,n}^{hpj} - L_{mk,n}^p a_{pl}^{hij} - L_{mk}^p a_{pl,n}^{hij} - \\ - L_{ml,n}^p a_{kp}^{hij} - L_{ml}^p a_{kp,n}^{hij} + L_{ns}^h a_{kl,m}^{sij} + L_{ns}^h L_{pm}^s a_{kl}^{pij} + L_{ns}^h L_{pm}^i a_{kl}^{spj} + \\ + L_{ns}^h L_{pm}^j a_{kl}^{sij} - L_{ns}^h L_{mk}^p a_{pl}^{hij} - L_{ns}^h L_{ml}^p a_{kp}^{hij} + L_{ns}^i a_{kl,m}^{hsj} + \\ + L_{ns}^i L_{pm}^h a_{kl}^{psj} + L_{ns}^i L_{pm}^s a_{hl}^{hpj} + L_{ns}^i L_{pm}^j a_{kl}^{hsp} - L_{ns}^i L_{mk}^p a_{pl}^{hsj} - \\ - L_{ns}^i L_{ml}^p a_{kp}^{hsj} + L_{ns}^j a_{kl,m}^{his} + L_{ns}^j L_{pm}^h a_{kl}^{pis} + L_{ns}^j L_{pm}^i a_{kl}^{hps} + \\ + L_{ns}^j L_{pm}^s a_{kl}^{hij} - L_{ns}^j L_{mk}^p a_{pl}^{his} - L_{ns}^j L_{ml}^p a_{kp}^{hij} - L_{kn}^s a_{sl,m}^{hij} - \\ - L_{kn}^s L_{pm}^h a_{sl}^{pij} - L_{kn}^s L_{pm}^i a_{sl}^{hpj} - L_{kn}^s L_{pm}^j a_{sl}^{hpj} + L_{kn}^s L_{ms}^p a_{pl}^{hij} + \\ + L_{kn}^s L_{ml}^p a_{sp}^{hij} - L_{ln}^s a_{ks,m}^{hij} - L_{ln}^s L_{pm}^h a_{ks}^{pij} - L_{ln}^s L_{pm}^i a_{ks}^{hpj} - \\ - L_{ln}^s L_{pm}^j a_{ks}^{hij} + L_{ln}^s L_{mk}^p a_{ps}^{hij} + L_{ln}^s L_{ms}^p a_{kp}^{hij} - L_{mn}^s a_{kl,s}^{hij},$$

from where we obtain

$$(16) \quad \begin{aligned} a_{kl|_m|_n}^{hij} - a_{kl|_n|_m}^{hij} &= A_{pmn}^h a_{kl}^{pij} + A_{pmn}^i a_{kl}^{hpj} + \\ &+ A_{pmn}^j a_{kl}^{hin} - A_{kmn}^p a_{pl}^{hij} - A_{lmn}^p a_{kp}^{hij} + \\ &+ 4 \tilde{a}_{kl \langle mn \rangle}^{hij} + 4 \tilde{a}_{kl \leq mn \geq}^{hij} - 2 L_{mn}^p a_{kl|_p}^{hij}, \end{aligned}$$

where we denoted

$$(17) \quad A_{jmn}^i = L_{jm,n}^i - L_{jm,n}^i + L_{jm}^p L_{np}^i - L_{jn}^p L_{mp}^i,$$

$$(18) \quad A_{jmn}^i = L_{mj,n}^i - L_{nj,m}^i + L_{jm}^p L_{np}^i - L_{jn}^p L_{mp}^i,$$

$$(19) \quad \tilde{a}_{kl \langle mn \rangle}^{hij} = L_{pm}^h a_{kl,n}^{pij} + L_{pm}^i a_{kl,n}^{hpj} + \dots - L_{mk}^p a_{pl,n}^{hij} - L_{ml}^p a_{kp,n}^{hij},$$

$$(20) \quad \tilde{a}_{kl \leq mn \geq}^{hij} = L_{[pm]}^h L_{[ns]}^i a_{kl}^{psj} + \dots - L_{[pm]}^h L_{[kn]}^s a_{sl}^{pij} - \dots + L_{[mk]}^p L_{[ln]}^s a_{ps}^{hij},$$

$$(21) \quad L_{[pm]}^h L_{[ns]}^i = \frac{1}{2} (L_{pm}^h L_{ns}^i - L_{mp}^h L_{sn}^i).$$

Generally, it is

$$(22) \quad \begin{aligned} a_{t_1 \dots t_v | m | n}^{r_1 \dots r_u} - a_{t_1 \dots t_v | n | m}^{r_1 \dots r_u} &= \\ &= \sum_{l=1}^u A_{pmn}^{r_l} \binom{p}{r_l} a_{\dots}^{r_1 \dots r_u} - \sum_{f=1}^v A_{tfmn}^p \binom{t_f}{p} a_{\dots}^{r_1 \dots r_u} + \\ &+ 4 \tilde{a}_{t_1 \dots t_v \langle mn \rangle}^{r_1 \dots r_u} + 4 \tilde{a}_{t_1 \dots t_v \leq mn \geq}^{r_1 \dots r_u} - 2 L_{mn}^p a_{t_1 \dots t_v | p}^{r_1 \dots r_u}, \end{aligned}$$

where

$$(23) \quad \tilde{a}_{t_1 \dots t_v \langle mn \rangle}^{r_1 \dots r_u} = \sum_{l=1}^u L_{pm}^{r_l} \binom{p}{r_l} a_{\dots, n}^{r_1 \dots r_u} - \sum_{f=1}^v L_{mtf}^p \binom{t_f}{p} a_{\dots, n}^{r_1 \dots r_u},$$

$$(24) \quad \begin{aligned} \tilde{a}_{t_1 \dots t_v \leq mn \geq}^{r_1 \dots r_u} &= \sum_{l=1}^{u-1} \sum_{f=2}^u L_{[pm]}^{r_l} L_{[ns]}^{r_f} \binom{p}{r_l} \binom{s}{r_f} a_{\dots}^{r_1 \dots r_u} - \\ &- \sum_{l=1}^u \sum_{f=1}^v L_{[pm]}^{r_l} L_{[tfm]}^s \binom{p}{r_l} \binom{t_f}{s} a_{\dots}^{r_1 \dots r_u} + \sum_{l=1}^{v-1} \sum_{f=2}^v L_{[mtl]}^p L_{[tfm]}^s \binom{t_l}{p} \binom{t_f}{s} a_{\dots}^{r_1 \dots r_u}. \end{aligned}$$

4. Applying two kinds of covariant derivation (3), (4) in inversed order than in preceding case, we obtain

$$\begin{aligned} (a_{kl|m})_{|n} &= (a_{kl|m})_{,n} + L_{sn}^h a_{kl}^{sij} + L_{sn}^i a_{kl|m}^{hsj} + L_{sn}^j a_{kl|m}^{his} - \\ &- L_{nk}^s a_{sl|m}^{hij} - L_{nl}^s a_{ks|m}^{hij} - L_{nm}^s a_{kl|s}^{hij}, \end{aligned}$$

i.e.

$$\begin{aligned}
 (25) \quad & a_{\substack{k \\ 4}}^{\substack{hij \\ kl}} \Big|_{m \mid n} = a_{kl, mn}^{hij} + L_{mp, n}^h a_{kl}^{pij} + L_{mp}^h a_{kl, n}^{pij} + L_{mp, n}^i a_{kl}^{hij} + \\
 & + L_{mp}^i a_{kl, n}^{hij} + L_{mp, n}^j a_{kl}^{hip} + L_{mp}^j a_{kl, n}^{hip} - L_{km, n}^p a_{pl}^{hij} - L_{km}^p a_{pl, n}^{hij} - \\
 & - L_{lm, n}^p a_{kp}^{hij} - L_{lm}^p a_{kp, n}^{hij} + L_{sn}^h a_{kl, m}^{sin} + L_{sn}^h L_{mp}^s a_{kl}^{pij} + L_{sn}^h L_{mp}^i a_{kl}^{spj} + \\
 & + L_{sn}^h L_{mp}^j a_{kl}^{spj} - L_{sn}^h L_{km}^p a_{pl}^{sij} - L_{sn}^h L_{lm}^p a_{kp}^{sij} + L_{sn}^i a_{kl, m}^{hsj} + L_{sn}^i L_{mp}^h a_{kl}^{psj} + \\
 & + L_{sn}^i L_{mp}^s a_{kl}^{hij} + L_{sn}^i L_{mp}^j a_{kl}^{hsp} - L_{sn}^i L_{km}^p a_{pl}^{hsj} - L_{sn}^i L_{lm}^p a_{kp}^{hsj} + \\
 & + L_{sn}^j a_{kl, m}^{his} + L_{sn}^j L_{mp}^h a_{kl}^{pis} + L_{sn}^j L_{mp}^i a_{kl}^{hps} + L_{sn}^j L_{mp}^s a_{kl}^{hip} - \\
 & - L_{sn}^j L_{km}^p a_{pl}^{his} - L_{sn}^j L_{lm}^p a_{kp}^{his} - L_{nk}^s a_{sl, m}^{hij} - L_{nk}^s L_{mp}^h a_{sl}^{pij} - \\
 & - L_{nk}^s L_{mp}^i a_{sl}^{hij} - L_{nk}^s L_{mp}^j a_{sl}^{hip} + L_{nk}^s L_{sm}^p a_{pl}^{hij} + L_{nk}^s L_{lm}^p a_{sp}^{hij} - \\
 & - L_{nl}^s a_{ks, m}^{hij} - L_{nl}^s L_{mp}^h a_{ks}^{pij} - L_{nl}^s L_{mp}^i a_{ks}^{hij} - L_{nl}^s L_{mp}^j a_{ks}^{hip} + \\
 & + L_{nl}^s L_{km}^p a_{ps}^{hij} + L_{nl}^s L_{sm}^p a_{kp}^{hij} - L_{nm}^s a_{kl, s}^{hij},
 \end{aligned}$$

from where it is

$$\begin{aligned}
 (26) \quad & a_{\substack{k \\ 4}}^{\substack{hij \\ kl}} \Big|_{m \mid n} - a_{\substack{k \\ 4}}^{\substack{hij \\ kl}} \Big|_{n \mid m} = A_{pmn}^h a_{kl}^{pij} + \dots - A_{kmn}^p a_{pl}^{hij} - \dots \\
 & - 4 \bar{a}_{kl \langle mn \rangle}^{hij} - 4 \bar{a}_{kl \leq mn \geq}^{hij} + 2 L_{mn}^p a_{kl \mid p}^{hij},
 \end{aligned}$$

where

$$(27) \quad A_{\substack{j \\ 3}}^i \Big|_{mn} = L_{mj, n}^i - L_{nj, m}^i + L_{mj}^n L_{pn}^i - L_{nj}^p L_{pm}^i,$$

$$(28) \quad A_{\substack{j \\ 2}}^i \Big|_{mn} = L_{jm, n}^i - L_{jn, m}^i + L_{mj}^p L_{pn}^i - L_{nj}^p L_{pm}^i.$$

Generally, it is

$$\begin{aligned}
 (29) \quad & a_{\substack{t_1 \dots t_v \\ 4}}^{\substack{r_1 \dots r_u \\ t_y}} \Big|_{m \mid n} - a_{\substack{t_1 \dots t_v \\ 4}}^{\substack{r_1 \dots r_u \\ t_y}} \Big|_{n \mid m} = \\
 & = \sum_{l=1}^u A_{\substack{r_l \\ 3}}^i \Big|_{pmn} \binom{p}{r_l} a \dots - \sum_{f=1}^v A_{\substack{r_f \\ 2}}^p \Big|_{fmn} \binom{t_f}{p} a \dots - \\
 & - 4 \bar{a}_{t_1 \dots t_v \langle mn \rangle}^{r_1 \dots r_u} - 4 \bar{a}_{t_1 \dots t_v \leq mn \geq}^{r_1 \dots r_u} + 2 L_{mn}^p a_{t_1 \dots t_v \mid p}^{r_1 \dots r_u}.
 \end{aligned}$$

5. Based on (3) and (12), we obtain

$$(30) \quad \begin{aligned} a_{kl|mn}^{hij} - a_{kl|m}^{hij} = & A_5^h p_{mn} a_{kl}^{pij} + A_5^i p_{mn} a_{kl}^{hpj} + A_5^j p_{mn} a_{kl}^{hip} + \\ & + A_6^p k_{nm} a_{pl}^{hij} + A_6^p l_{nm} a_{kp}^{hij} + \\ & + 4 \bar{a}_{kl\langle mn\rangle}^{hij} + 4 \bar{a}_{kl\leq mn}^{hij} - L_{mn}^p (a_{kl|p}^{hij} - a_{kl|p}^{hij}), \end{aligned}$$

where

$$(31) \quad A_5^i j_{mn} = L_{jm,n}^i - L_{nj,m}^i + L_{jm}^p L_{pn}^i - L_{nj}^p L_{mp}^i,$$

$$(32) \quad A_6^i j_{mn} = L_{jm,n}^i - L_{nj,m}^i + L_{mj}^p L_{np}^i - L_{jn}^p L_{pm}^i,$$

$$(33) \quad \bar{a}_{kl\leq mn}^{hij} = L_{[pm]}^h L_{[sn]}^i a_{kl}^{psj} + \dots - L_{[pm]}^h L_{[nk]}^s a_{sl}^{pji} - \dots + L_{[mk]}^p L_{[nl]}^s a_{ps}^{hij},$$

and \underline{mn} means the symmetrisation on the indices m, n .

Generally, we have

$$(34) \quad \begin{aligned} a_{t_1 \dots t_v | mn}^{r_1 \dots r_u} - a_{t_1 \dots t_v | nm}^{r_1 \dots r_u} = & \\ = & \sum_{l=1}^u A_5^{r_l} p_{mn} \binom{p}{r_l} a \dots + \sum_{f=1}^v A_6^p t_f nm \binom{t_f}{p} a \dots + \\ & + 4 \bar{a}_{t_1 \dots t_v \langle mn \rangle}^{r_1 \dots r_u} + 4 \bar{a}_{t_1 \dots t_v \leq mn}^{r_1 \dots r_u} - L_{mn}^p (a_{t_1 \dots t_v | p}^{r_1 \dots r_u} - a_{t_1 \dots t_v | p}^{r_1 \dots r_u}), \end{aligned}$$

where, analogically to (24), we put

$$(35) \quad \begin{aligned} \bar{a}_{t_1 \dots t_v \leq mn}^{r_1 \dots r_u} = & \sum_{l=1}^{u-1} \sum_{f=1}^u L_{[pm]}^{r_l} L_{[sn]}^{r_f} \binom{p}{r_l} \binom{s}{r_f} a \dots - \\ - & \sum_{l=1}^u \sum_{f=1}^v L_{[pm]}^{r_l} L_{[ntf]}^s \binom{p}{r_l} \binom{t_f}{s} a \dots + \sum_{l=1}^{u-1} \sum_{f=2}^v L_{[mt]}^p L_{[ntf]}^s \binom{t_l}{p} \binom{t_f}{s} a \dots . \end{aligned}$$

6. Using (5), (15), we obtain

$$(36) \quad \begin{aligned} a_{kl|mn}^{hij} - a_{kl|m}^{hij} = & A_7^h p_{mn} a_{kl}^{pij} + \dots - A_{14}^p k_{mn} a_{pl}^{hij} - \dots \\ & + 2 \bar{a}_{kl\langle mn \rangle}^{hij} + 2 \bar{a}_{kl\leq mn}^{hij}, \end{aligned}$$

where

$$(37) \quad A_{\frac{7}{7}}^j{}_{jmn} = L_{jm,n}^i - L_{jn,m}^i + L_{jm}^p L_{pn}^i - L_{jn}^p L_{mp}^i,$$

$$(38) \quad A_{\frac{14}{14}}^i{}_{jmn} = L_{mj,n}^i - L_{nj,m}^i + L_{jm}^p L_{np}^i - L_{nj}^p L_{mp}^i,$$

$$(39) \quad \bar{a}_{kl \leq mn \geq}^{hij} = a_{kl}^{psj} (\underbrace{L_{pm}^h L_{sn}^i}_{\vee} + \underbrace{L_{pn}^h L_{sm}^i}_{\vee}) + \dots$$

$$- a_{sl}^{pjj} (\underbrace{L_{pm}^h L_{nk}^s}_{\vee} + \underbrace{L_{pn}^h L_{mk}^s}_{\vee}) - \dots + a_{ps}^{hij} (\underbrace{L_{mk}^p L_{nl}^s}_{\vee} + \underbrace{L_{nk}^p L_{ml}^s}_{\vee}).$$

Generally

$$(40) \quad a_{t_1 \dots t_v \frac{3}{3} mn}^{r_1 \dots r_u} - a_{t_1 \dots t_v \frac{3}{4} n \frac{1}{1} m}^{r_1 \dots r_u} = \\ + \sum_{l=1}^u A_{\frac{7}{7}}^{r_l}{}_{pmn} \binom{p}{r_l} a \dots - \sum_{f=1}^v A_{\frac{14}{14}}^p{}_{tfmn} \binom{t_f}{p} a \dots + \\ + 2 \bar{a}_{t_1 \dots t_v \langle mn \rangle}^{r_1 \dots r_u} + 2 \bar{a}_{t_1 \dots t_v \leq mn \geq}^{r_1 \dots r_u},$$

where

$$(41) \quad a_{t_1 \dots t_v \leq mn \geq}^{r_1 \dots r_u} = \sum_{l=1}^{u-1} \sum_{f=2}^u (L_{pm}^{r_l} L_{sn}^{r_f} + L_{pn}^{r_l} L_{sm}^{r_f}) \binom{p}{r_l} \binom{s}{r_f} a \dots - \\ - \sum_{l=1}^u \sum_{f=1}^v (L_{pm}^{r_l} L_{nt_f}^s + L_{pn}^{r_l} L_{mt_f}^s) \binom{p}{r_l} \binom{t_f}{s} a \dots + \\ + \sum_{l=1}^{v-1} \sum_{f=2}^v (L_{mt_l}^p L_{nt_f}^s + L_{nt_l}^p L_{mt_f}^s) \binom{t_l}{p} \binom{t_f}{s} a \dots.$$

7. From (5) and (25) one obtains

$$(42) \quad a_{\frac{3}{3} kl \mid mn}^{hij} - a_{\frac{4}{4} kl \mid n \mid m}^{hij} = A_{\frac{9}{9}}^h{}_{pmn} a_{kl}^{pjj} + \dots - A_{\frac{12}{12}}^p{}_{kmn} a_{pl}^{hij} - \dots \\ + 2 \bar{a}_{kl \langle nm \rangle}^{hij} + 2 \bar{a}_{kl \leq nm \geq}^{hij} - (L_{nm}^p a_{kl \mid p}^{hij} - L_{mn}^p a_{kl \mid p}^{hij}),$$

where we have put

$$(43) \quad A_{\frac{9}{9}}^i{}_{jmn} = L_{jm,n}^i - L_{nj,m}^i + L_{jm}^p L_{pn}^i - L_{nj}^p L_{pm}^i,$$

$$(44) \quad A_{\frac{12}{12}}^i{}_{jmn} = L_{mj,n}^i - L_{jn,m}^i + L_{mj}^p L_{pn}^i - L_{nj}^p L_{mp}^i.$$

In general

$$(45) \quad a_{\frac{r_1 \dots r_u}{3} mn}^{r_1 \dots r_u} - a_{\frac{r_1 \dots r_u}{4} n|m}^{r_1 \dots r_u} =$$

$$= \sum_{l=1}^u A_{pmn}^r \binom{p}{r_l} a \dots - \sum_{f=1}^v A_{tfmn}^p \binom{t_f}{p} a \dots +$$

$$+ 2 \tilde{a}_{t_1 \dots t_v \langle nm \rangle}^{r_1 \dots r_u} + 2 \tilde{a}_{t_1 \dots t_v \leq mn \geq}^{r_1 \dots r_u} - (L_{nm}^p a_{\frac{r_1 \dots r_u}{3} p}^{r_1 \dots r_u} - L_{mn}^p a_{\frac{r_1 \dots r_u}{4} p}^{r_1 \dots r_u}).$$

8. On the base of (12) and (13):

$$(46) \quad a_{\frac{hij}{4} mn}^{hij} - a_{\frac{hij}{3} n|m}^{hij} = A_{pmn}^h a_{kl}^{pji} + \dots$$

$$- A_{kmn}^p a_{pl}^{hij} - \dots - 2 \tilde{a}_{kl \langle nm \rangle}^{hij} + 2 \tilde{a}_{kl \leq mn \geq}^{hij} + (L_{nm}^p a_{\frac{hij}{3} p}^{hij} - L_{mn}^p a_{\frac{hij}{4} p}^{hij}),$$

where

$$(47) \quad A_{jmn}^i = L_{mj,n}^i - L_{jn,m}^i + L_{mj}^p L_{np}^i - L_{jn}^p L_{mp}^i,$$

$$(48) \quad A_{jm n}^i = L_{jm,n}^i - L_{nj,m}^i + L_{jm}^p L_{np}^i - L_{jn}^p L_{pm}^i,$$

$$(49) \quad \tilde{a}_{kl \leq mn \geq}^{hij} = a_{ki}^{psj} (L_{mp}^h L_{ns}^i + L_{np}^h L_{ms}^i) + \dots$$

$$- a_{sl}^{pji} (L_{mp}^h L_{kn}^s + L_{np}^h L_{km}^s) - \dots + a_{ps}^{hij} (L_{km}^p L_{ln}^s + L_{kn}^p L_{lm}^s).$$

Generally, it is

$$(50) \quad a_{\frac{r_1 \dots r_u}{4} mn}^{r_1 \dots r_u} - a_{\frac{r_1 \dots r_u}{3} n|m}^{r_1 \dots r_u} =$$

$$= \sum_{l=1}^u A_{pmn}^r \binom{p}{r_l} a \dots - \sum_{f=1}^v A_{tfmn}^p \binom{t_f}{p} a \dots -$$

$$- 2 \tilde{a}_{t_1 \dots t_v \langle nm \rangle}^{r_1 \dots r_u} + 2 \tilde{a}_{t_1 \dots t_v \leq mn \geq}^{r_1 \dots r_u} + (L_{nm}^p a_{\frac{r_1 \dots r_u}{3} p}^{r_1 \dots r_u} - L_{mn}^p a_{\frac{r_1 \dots r_u}{4} p}^{r_1 \dots r_u}),$$

where

$$(51) \quad \tilde{a}_{t_1 \dots t_v \leq mn \geq}^{r_1 \dots r_u} = \sum_{l=1}^{u-1} \sum_{f=2}^u (L_{mp}^{r_l} L_{ns}^{r_f} + L_{np}^{r_l} L_{ms}^{r_f}) \binom{p}{r_l} \binom{s}{r_f} a \dots -$$

$$- \sum_{l=1}^u \sum_{f=1}^v (L_{mp}^{r_l} L_{fn}^s + L_{np}^{r_l} L_{fm}^s) \binom{p}{r_l} \binom{t_f}{s} a \dots +$$

$$+ \sum_{l=1}^{v-1} \sum_{f=2}^v (L_{tm}^p L_{ifn}^s + L_{tn}^p L_{ifm}^s) \binom{t_l}{p} \binom{t_f}{s} a \dots .$$

9. By (12) and (25) we get

$$(52) \quad \begin{aligned} a_{kl|mn}^{hij} - a_{kl|n|m}^{hij} &= A_{pmn}^h a_{kl}^{pij} + \dots \\ &\quad - A_{knn}^p a_{pl}^{hij} - \dots - 2\bar{a}_{kl<mn>}^{hij} + 2\bar{a}_{kl<nm>}^{hij}, \end{aligned}$$

where it is

$$(53) \quad A_{jmnn}^i = L_{mj,n}^i - L_{nj,m}^i + L_{mj}^p L_{np}^i - L_{nj}^p L_{pm}^i,$$

$$(54) \quad A_{jmnn}^i = L_{jm,n}^i - L_{jn,m}^i + L_{mj}^p L_{pn}^i - L_{jn}^p L_{pm}^i.$$

Generally

$$(55) \quad \begin{aligned} a_{t_1 \dots t_v | mn}^{r_1 \dots r_u} - a_{t_1 \dots t_v | n|m}^{r_1 \dots r_u} &= \\ &= \sum_{l=1}^u A_{pnm}^{rl} \binom{p}{r_l} a \dots - \sum_{f=1}^v A_{tfmn}^p \binom{t_f}{p} a \dots - \\ &\quad - 2\bar{a}_{t_1 \dots t_v < mn >}^{r_1 \dots r_u} + 2\bar{a}_{t_1 \dots t_v < nm >}^{r_1 \dots r_u}. \end{aligned}$$

10. In the end, by (15) and (25), we obtain

$$(56) \quad \begin{aligned} a_{kl|m|n}^{hij} - a_{kl|n|m}^{hij} &= A_{pmn}^h a_{kl}^{pij} + A_{pmn}^i a_{kl}^{hpi} + A_{pmn}^j a_{kl}^{hip} + \\ &\quad + A_{knn}^p a_{pl}^{hij} + A_{lnm}^p a_{kp}^{hij} - L_{mn}^p (a_{kl|p}^{hij} - a_{kl|p}^{hij}), \end{aligned}$$

where we designated

$$(57) \quad A_{jmnn}^i = L_{jm,n}^i - L_{nj,m}^i + L_{jm}^p L_{np}^i - L_{nj}^p L_{pm}^i.$$

Generally, it is

$$(58) \quad \begin{aligned} a_{t_1 \dots t_v | m|n}^{r_1 \dots r_u} - a_{t_1 \dots t_v | n|m}^{r_1 \dots r_u} &= \sum_{l=1}^u A_{pnm}^{rl} \binom{p}{r_l} a \dots + \\ &\quad + \sum_{f=1}^v A_{tfnm}^p \binom{t_f}{p} a \dots - L_{mn}^p (a_{t_1 \dots t_v | p}^{r_1 \dots r_u} - a_{t_1 \dots t_v | p}^{r_1 \dots r_u}). \end{aligned}$$

The magnitudes A_1, \dots, A_{15} , which we introduced for the first time in the work [1], are not tensors, but because they have a form and a role of curvature tensors, we called them in [1] *curvature pseudotensors* of the non symmetric affine connexion space L_N .

The equation (56) can be transformed into another form. In fact, if we count the difference in the right side in (56) according to (3), (4), we obtain

$$(56') \quad \begin{aligned} a_{kl|m|n}^{hij} - a_{kl|n|m}^{hij} &= R_{pmn}^h a_{kl}^{pij} + R_{pmn}^i a_{kl}^{hpi} + \\ &\quad + R_{pmn}^j a_{kl}^{hip} + R_{knn}^p a_{pl}^{hij} + R_{lnm}^p a_{kp}^{hij}, \end{aligned}$$

where we put

$$(59) \quad R^i_{jmn} = \underset{15}{A^i_{jmn}} + L^p_{nm} (L^i_{pj} - L^i_{jp}) = \\ = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm} L^i_{np} - L^p_{nj} L^i_{pm} + L^p_{nm} (L^i_{pj} - L^i_{jp}),$$

$$(60) \quad R^i_{jmn} = \underset{4}{A^i_{jmn}} + L^p_{mn} (L^i_{pj} - L^i_{jp}) = \\ = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm} L^i_{np} - L^p_{nj} L^i_{pm} + L^p_{mn} (L^i_{pj} - L^i_{jp}).$$

Generally, the following is valid

$$(58') \quad a^{r_1 \dots r_u}_{t_1 \dots t_v | m | n} - a^{r_1 \dots r_u}_{t_1 \dots t_v | n | m} = \\ = \sum_{l=1}^u R^{r_l}_{pnm} \binom{p}{r_l} a \dots = \sum_{f=1}^v R^p_{tfnm} \binom{t_f}{p} a \dots$$

The magnitudes R and R are tensors and we call them *curvature tensors* of the 3rd respectively 4th kind of the space L_N .

The identities (6) and (10), (13) and (14), (16) and (22), (26) and (29), (30) and (34), (36) and (40) (42) and (45), (46) and (50), (52) and (55), (56) and (58), (56') and (58') are *generalizations of the Ricci identity* in the symmetric affine connexion space.

Also, all the curvature tensors R , R , R , R and all the curvature pseudo-tensors A , \dots , A are *generalizations of the Riemann — Cristoffel curvature tensor* R^i_{jmn} and become the same tensor in the symmetric space.

R E F E R E N C E

- [1] M i n č ić, S. M., *Ricci identities in the space of nonsymmetric affine connexion*, Matematički vesnik, 10 (25), 1975, Beograd, 161—172.

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