PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 22 (36), 1977, pp. 175—185

RELATIVISTIC RELATIVE DEFORMATION AND VORTICITY APPLIED TO MAGNETOHYDRODYNAMICS

I. Lukačević

(Communicated March 2, 1977)

This paper is related to the same subject as [11], with applications to *MHD*. Some of previous results are completed and an error corrected.

The first part of the paper is limited to kinematics, deformation being formulated by means of two congruences of timelike curves in spacetime. One of these congruences represents the stream lines of a material continuum, the other one can be understood as the system of world lines corresponding to the proper time of a galileian observer in Special Relativity, or to the coordinate time in General Relativity. These limitations are given only as examples. Several particular cases are then considered, for which relative deformation becomes purely spatial, and two or three dimensional; relations are obtained which "propagate" that state of deformation in spacetime.

In the second part an application is made, of the preceding, to a *MHD* fluid, given only through its stream lines and its magnetic field. Then a vorticity vector is also considered, coupled with the electric current. We point out the fact that no tensor of energy is used. Some properties of quadratic first invariants are obtained in the third part.

Some of the papers given in the references are not explicitly quoted in the text, but they were in our previous papers, or are generally related to the same subject.

*

* *

Relative deformation, or quasideformation (as we called it throughout the preceding paper [11], in a material continuum given only through a congruence of timelike streamlines, with tangent unit four velocities u^α i spacetime V_4 of signature $(+, +, +, -)$, was defined simply by the Lie (or convective) derivative of the projecting tensor $h_{\alpha\beta}$ of that continuum, with respect to another congruence of timelike curves, with unit four velocities ξ^α:

$$\begin{aligned}(1.1)\qquad \nu_{\alpha\beta} &= \mathcal{L}_\xi h_{\alpha\beta} = \mathcal{L}_\xi (g_{\alpha\beta} + u_\alpha u_\beta) = \\ &= \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha + u_\alpha \xi^\gamma \nabla_\gamma u_\beta + u_\beta \xi^\gamma \nabla_\gamma u_\alpha + u_\alpha u^\gamma \nabla_\beta \xi_\gamma + u_\beta u^\gamma \nabla_\alpha \xi_\gamma .\end{aligned}$$

All the considered quantities must be at least second order derivable. Let us remark immediately that for $\xi^\alpha = u^\alpha$ obtained tensor differs slightly from the traceless shear tensor (cf Ehlers [4]) $\sigma_{\alpha\beta}$ which has an additional term $-\frac{1}{3}\nabla_\gamma u^\gamma h_{\alpha\beta}$, but this is not essential, as was already pointed out in [11]. The proper deformation tensor, without that additional term, represents, in fact, the departure from Born's rigidity.

1) We shall make next assumption concerning relative deformation: Tensor $\nu_{\alpha\beta}$ allows a real timelike eigenvector ζ^α. It must have, therefrom, not only another real eigenvalue, but all the remaining three, with corresponding spacelike eigenvectors. That is the consequence of the symmetry of $\nu_{\alpha\beta}$ and the definiteness of spacelike metric (cf Synge [2]).

Tensor $\nu_{\alpha\beta}$ satisfies identically the condition:

$$\nu_{\alpha\beta} u^\alpha u^\beta = 0. \tag{1.2}$$

As a consequence of assumption 1) we can reduce it, with respect to its local principal frame, to diagonal form $\nu_{\alpha\beta}$ (with ν_{ii} for spacelike axes, and ν_{44} for timelike one). If we denote the eigenvectors of that tensor by $\zeta^\alpha_{(\gamma)}\,(\gamma = 1, 2, 3, 4)$, the fourth being timelike:

$$\nu_{\alpha\beta}\,\zeta^\beta_{(\gamma)} = \lambda_{(\gamma)}\,\zeta_{(\gamma)\alpha}, \tag{1.3}$$

we shall have, in the principal frame:

$$\nu_{ii} = \lambda_{(i)}, \quad \nu_{44} = -\lambda_{(4)}.$$

Therefrom:

$$\nu_{11}(u^1)^2 + \nu_{22}(u^2)^2 + \nu_{33}(u^3)^2 + \nu_{44}(u^4)^2 = 0. \tag{1.2'}$$

The above quadratic form is satisfied by values of u^α symmetric with respect to its $\zeta^\alpha_{(4)}$ principal axis, being so timelike. So that local hypercone has not only one timelike axis, but essentially timelike directions among its generatrices. In the special case when all the vectors ϑ^α, satisfying a relation of the (1.2′) form, are timelike, that hypercone is directed by a spacelike ellipsoid. Its half-cones contain corresponding branches of hyperbolae for appropriate values of constants

$$\nu_{\alpha\beta}\,\vartheta^\alpha\,\vartheta^\beta = \text{const}$$

and there are external hyperbolae for values of constants with opposite sign, the two families being separated by asymptotic hypersurface (1.2′). It results from relation (1.2) that under condition 1) a cone with real axes exists, containing necessarily timelike vectors, among which u^α, but with the possibility of other kinds of generatrices. The same holds for the families of hyperbolae.

We considered in [11] also a spacelike tensor $\tau_{\alpha\beta}$, defined by:

$$\begin{aligned} \tau_{\alpha\beta} &= \nu_{\gamma\delta} h^\gamma_\alpha h^\delta_\beta = (\nabla_\gamma \xi_\delta + \nabla_\delta \xi_\gamma) h^\gamma_\alpha h^\delta_\beta = \\ &= \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha + u_\alpha u^\gamma \nabla_\gamma \xi_\beta + u_\beta u^\gamma \nabla_\gamma \xi_\alpha + u_\alpha u^\gamma \nabla_\beta \xi_\gamma + \\ &\quad + u_\beta u^\gamma \nabla_\alpha \xi_\gamma + 2 u_\alpha u_\beta u^\gamma u^\delta \nabla_\gamma \xi_\delta. \end{aligned} \tag{1.4}$$

Let us form the above projections in the local principal frame of $\nu_{\alpha\beta}$. These are:

$$\tau_{ii}=\nu_{ii}\,[1+2\,(u_i)^2],\quad \tau_{44}=\nu_{44}\,[1-2\,(u_4)^2],$$
$$\tau_{ij}=u_i\,u_j\,(\nu_{ii}+\nu_{jj}),\quad \tau_{i4}=u_i\,u_4\,(\nu_{ii}-\nu_{44}). \tag{1.4'}$$

We see that diagonal terms of $\tau_{\alpha\beta}$ can vanish only simultaneously with $\nu_{\alpha\alpha}$. Therefrom:

$$\sum_{\alpha=1}^{4}\frac{\tau_{\alpha\alpha}}{\nu_{\alpha\alpha}}=2, \tag{1.5}$$

$$u_1\,\tau_{23}+u_2\,\tau_{31}+u_3\,\tau_{12}-2\,u_1\,u_2\,u_3\,(\nu_{11}+\nu_{22}+\nu_{33}), \tag{1.6}$$

and

$$u_1\,u_2\,\tau_{34}+u_3\,u_1\,\tau_{24}+u_2\,u_3\,\tau_{14}=u_1\,u_2\,u_3\,u_4\,(\nu_{11}+\nu_{22}+\nu_{33}-3\,\nu_{44}). \tag{1.7}$$

Expression (1.5) represents, under the assumption of non vanishing canonical components of $\nu_{\alpha\beta}$, a relation between $\tau_{\alpha\alpha}$ and $\nu_{\alpha\alpha}$ independent of four velocity u_α; meanwhile (1.6), (1.7) express the trace of $\nu_{\alpha\beta}$ in function of u_α and $\tau_{\alpha\beta}$.

We shall assume now the nullity of the quadritic forms $\nu_{\alpha\beta}\,\xi^\alpha\,\xi^\beta$ and $\nu_{\alpha\beta}\,u^\alpha\,\xi^\beta$. So:

$$\text{a)}\quad \nu_{\alpha\beta}\,\xi^\alpha\,\xi^\beta=0 \Leftrightarrow \xi^\alpha\,\partial_\alpha\,\vartheta=0,\quad (\theta=g_{\alpha\beta}\,u^\alpha\,\xi^\beta<-1) \tag{1.8}$$

and

$$\text{b)}\quad \nu_{\alpha\beta}\,u^\alpha\,\xi^\beta=0 \Leftrightarrow \xi^\alpha\,\xi^\beta\,\nabla_\alpha\,u_\beta-\theta\,u^\alpha\,u^\beta\,\nabla_\alpha\,\xi_\beta=0. \tag{1.9}$$

Above relations can be verified at once. We shall draw the consequences of a), in analogy with (1.4), (1.4'), forming:

$$\rho_{\alpha\beta}=\nu_{\gamma\delta}\,(\delta^\gamma_\alpha+\xi_\alpha\,\xi^\gamma)\,(\delta^\delta_\beta+\xi_\beta\,\xi^\delta). \tag{1.10}$$

Using (1.8) we obtain:

$$\rho_{\alpha\beta}=\sigma'_{\alpha\beta}+(u_\alpha+\vartheta\xi_\alpha)\,\mathscr{L}_\xi\,u_\beta+(u_\beta+\theta\xi_\beta)\,\mathscr{L}_\xi\,u_\alpha, \tag{1.11}$$

with

$$\sigma'_{\alpha\beta}=\nabla_\alpha\,\xi_\beta+\nabla_\beta\,\xi_\alpha+\xi_\alpha\,\xi^\gamma\,\nabla_\gamma\,\xi_\beta+\xi_\beta\,\xi^\gamma\,\nabla_\gamma\,\xi_\alpha.$$

Tensor $\sigma'_{\alpha\beta}$ expresses the departures from Born's rigidity in the field ξ^α. We should obtain relations of the form (1.4') between $\rho_{\alpha\beta}$, ξ^α and $\nu_{\alpha\beta}$, this being reduced to its principal frame, and then consequences of the form (1.5), (1.6), (1.7).

With b) the procedure would be the same. But then a nonsymmetric tensor $\varkappa_{\alpha\beta}$ appears:

$$\varkappa_{\alpha\beta}=\nu_{\gamma\beta}\,(\delta^\gamma_\alpha+u_\alpha\,u^\gamma)\,(\delta^\delta_\beta+u_\beta\,u^\delta). \tag{1.12}$$

When writing down diagonal relations analogous to (1.4'):

$$\varkappa_{ii}=\nu_{ii}\,[1+(u_i)^2+(\xi_i)^2],\quad \varkappa_{44}=\nu_{44}\,[1-(u_i)^2-(\xi_i)^2],$$

we obtain, from a) and b), for $\rho_{\alpha\beta}$ and $\varkappa_{\alpha\beta}$, simple relations analogous to (1.5):

$$\sum_{\alpha=1}^{4} \frac{\rho_{\alpha\alpha}}{\nu_{\alpha\alpha}} = \sum_{\alpha=1}^{4} \frac{\varkappa_{\alpha\alpha}}{\nu_{\alpha\alpha}} = 2. \tag{1.13}$$

Other relations corresponding to (1.6) and (1.7) are simple to write. The conclusions is:

The components of a normal relative deformation tensor $\nu_{\alpha\beta}$ (a tensor with a real timelike eigenvector), when expressed in its principal frame, are connected with the components of tensor $\tau_{\alpha\beta}$, given by (1.4), *by relations* (1.5), (1.6) *and* (1.7). *Under conditions a) and b) tensors $\rho_{\alpha\beta}$ and $\varkappa_{\alpha\beta}$, analogous to $\tau_{\alpha\beta}$, are related in the same way to $\nu_{\alpha\beta}$ by* (1.13), *and other expressions corresponding to* (1.6) *and* (1.7).

We have in general for $\nu_{\alpha\beta}$:

$$\begin{aligned} &\nu^{\alpha}_{\beta}\, u^{\beta} = \mathcal{L}_u\, \xi^{\alpha} + \varphi\, u^{\alpha}, \qquad (\varphi = u^{\beta} u^{\gamma} \nabla_{\beta} \xi_{\gamma}) \\ &\nu_{\alpha\beta}\, \xi^{\beta} = \mathcal{L}_{\xi} (\xi_{\alpha} + \vartheta\, u_{\alpha}). \end{aligned} \tag{1.14}$$

We shall now take into consideration relations a) and b), and add them assumption 2), not taking into consideration 1).

2) Vectors u^{α} and ξ^{α} define a local timelike 2-flat Π; we assume that one of the eigenvectors, say $\zeta^{\alpha}_{(1)}$, of $\nu_{\alpha\beta}$, lies in the local spacelike 2-flat Π^*, which is the orthogonal complement of Π.

We have then:

$$\begin{aligned} &\nu_{\alpha\beta}\, u^{\beta} = \alpha\, u_{\alpha} + \beta \xi_{\alpha} + \gamma\, l_{\alpha}, \\ &\nu_{\alpha\beta}\, \xi^{\beta} - \alpha'\, u_{\alpha} + \beta'\, \xi_{\alpha} + \gamma'\, l_{\alpha}, \end{aligned} \tag{1.14'}$$

where l_{α} is a vector of Π^*, orthogonal to $\zeta^{\alpha}_{(1)}$. Multiplying (1.14′) successively by u^{α}, ξ^{α}, on account of the identity (1.2) and relations a) and b) (eqs (1.8), (1.9)), terms with l_{α} vanish and right hand sides of these relations become homogeneous in α, β, α', β' with non vanishing determinants. So these coefficients are null. Denoting by $-\lambda$ the ratio γ/γ' ($\gamma' \neq 0$) we obtain:

$$\nu_{\alpha\beta} (u^{\beta} + \lambda \xi^{\beta}) = 0. \tag{1.15}$$

The conclusion is that under assumption 2), *i.e. orthogonality to u^{α} and ξ^{α} of one eigenvector of $\nu_{\alpha\beta}$, and relations a) and b), which imply that the pseudoangle between u^{α} and ξ^{α} remains constant along each ξ^{α} world line (but possibly variable from one to other), and that the ratio between projections $\xi^{\alpha} \xi^{\beta} \nabla_{\alpha} u_{\beta}$ and $u^{\alpha} u^{\beta} \nabla_{\alpha} \xi_{\beta}$ is equal to θ ($= \xi^{\alpha} u_{\alpha}$), relative deformation tensor $\nu_{\alpha\beta}$ has no components in the direction $u^{\alpha} + \lambda \xi^{\alpha}$, which can be of any type in spacetime.*

It is easy to verify that our condition is sufficient and obtain, from (1.15), relations a) and b).

Let us remark that the nullity of two quadratic and one bilinear from, i.e. identity (1.2) and conditions a) and b) only, lead to the transformation of Π in Π^*, which becomes, with assumption 2), singular, mapping Π into the straight line defined by l^{α}.

Consider the case when Π is an invariant tangent subspace with respect to $\nu_{\alpha\beta}$:

$$(1.16)\qquad \begin{aligned} \nu_{\alpha\beta}(au^\beta + b\xi^\beta) &= cu_\alpha + d\xi_\alpha, \\ \nu_{\alpha\beta}(a' u^\beta + b'\xi^\beta) &= c' u_\alpha + d'\xi_\alpha. \end{aligned}$$

Since we have, from definition (1.1):

$$\nu^\gamma_\beta u^\beta = -(\delta^\gamma_\beta + u^\gamma u_\beta)\mathscr{L}_\xi u^\beta,$$

then combining with (1.16), one obtains:

$$(1.17)\qquad \mathscr{L}_\xi u^\gamma = \lambda u^\gamma + \mu\xi^\gamma.$$

i.e. conditions (1.16) are sufficient for fields u^α and ξ^α to be surface forming. We remark that conditions for $\nu_{\alpha\beta}$ to have eigenvectors in Π are sufficient for (1.16). Let us take them as an assumption.

3) $\nu_{\alpha\beta}$ has two real eigenvectors in the local 2-flat Π. We shall add, to that assumption, first relation a) only, and examine proceeding consequences.

Since we have then

$$(1.17')\qquad \begin{aligned} \nu_{\alpha\beta}(au^\beta + b\xi^\beta) &= \varkappa_{(3)}(au_\alpha + b\xi_\alpha), \\ \nu_{\alpha\beta}(a' u^\beta + b'\xi^\beta) &= \varkappa_{(4)}(a' u_\alpha + b'\xi_\alpha), \end{aligned}$$

relations (1.2) and a) give us successively:

$$\begin{aligned} \varkappa_{(3)} &= 0 \ \vee\ a = \pm b, \\ \varkappa_{(4)} &= 0 \ \vee\ a' = \pm b. \end{aligned}$$

So for eigenvalues $\varkappa_{(3)}$, $\varkappa_{(4)}$ different from zero, eigenvector $\zeta^\alpha_{(3)}$ is spacelike, $\zeta^\alpha_{(4)}$ timelike:

$$(1.18)\qquad \zeta^\alpha_{(3)} = u^\alpha - \xi^\alpha, \quad \zeta^\alpha_{(4)} = u^\alpha + \xi^\alpha, \quad \varkappa_{(3)} = \varkappa_{(4)}\frac{\theta - 1}{\theta + 1},$$

and

$$(1.18')\qquad \begin{aligned} \varkappa_{(4)} > 0 &\Rightarrow \varkappa_{(3)} > \varkappa_{(4)}, \\ \varkappa_{(4)} = 0 &\Rightarrow \varkappa_{(3)} = 0, \\ \varkappa_{(4)} < 0 &\Rightarrow \varkappa_{(3)} < \varkappa_{(4)}. \end{aligned}$$

The first two conlusions (1.18) were drawn in [11] under the same assumptions, but we obtained also that then $\vartheta = -2$, which is an error, the only condition being $\vartheta < -1$.

We obtain, when adding relation b) to previous ones:

$$(1.19)\qquad \nu_{\alpha\beta}u^\beta = \nu_{\alpha\beta}\xi^\beta = 0.$$

Under condition 3) *and relations a*), *b*), $\nu_{\alpha\beta}$ *has no components in* Π.

Having considered these several cases, we shall choose among them to draw more conclusions, not considering only local algebraic properties.

Let us take the case of u^α, ξ^α forming a two parameter family of 2-surfaces S in spacetime with a $\nu_{\alpha\beta}$ satisfying relation (1.15) (with $\lambda \neq 0$). That condition reads explicitly:

$$\nu_{\alpha\beta}(u^\beta + \lambda\xi^\beta) = g_{\alpha\beta}\mathcal{L}_u \xi^\beta - u_\alpha u^\beta u^\gamma \nabla_\beta \xi_\gamma + \lambda_\xi(\xi_\alpha + \delta u_\alpha) = 0. \tag{1.20}$$

The condition of forming surfaces requires, from the above, that:

$$\mathcal{L}_\xi(\xi_\alpha + \vartheta u_\alpha) = \alpha u_\alpha + \beta\xi_\alpha. \tag{1.21}$$

After multiplication by u^α and ξ^α, having in mind that a) and b) are the consequences of (1.20), we obtain first $\beta = \alpha\vartheta$, then ϑ must be of unit modulus. This being contradictory, α and β must be null. So $\nu_{\alpha\beta}$ has no components in Π, and from (1.14):

$$\begin{aligned} &\mathcal{L}_\xi u^\alpha - \varphi u^\alpha = 0, \quad (\varphi = u^\beta u^\gamma \nabla_\beta \xi_\gamma), \\ &\mathcal{L}_\xi(\xi_\alpha + \theta u_\alpha) = 0. \end{aligned} \tag{1.22}$$

The first relation (1.22) expresses the fact that $\nu_{\alpha\beta}$ has no components in direction u^α. It reduces therefrom to its projection, orthogonal to u^α, given by $\tau_{\alpha\beta}$ in (1.4):

$$\nu_{\alpha\beta} = \tau_{\alpha\beta} = \gamma_{\gamma\delta}(\delta^\gamma_\alpha + u_\alpha u^\gamma)\ (\delta^\delta_\beta + u_\beta u^\delta).$$

The second relation (1.22) reduces $\nu_{\alpha\beta}$ to its projection orthogonal with respect to 2-flat Π. These vectors being timelike we have:

$$\nu_{\alpha\beta} = \nu_{\gamma\delta}\, t^\gamma_\alpha t^\delta_\beta \tag{1.23}$$

where

$$t_{\alpha\beta} = g_{\alpha\beta} - \frac{1}{\theta^2 - 1}(u_\alpha u_\beta + \vartheta u_\alpha \xi_\beta + \vartheta\xi_\alpha u_\beta + \xi_\alpha \xi_\beta).$$

"Biprojecting" tensor $t_{\alpha\beta}$ is analogous the one introduced in (Greenberg [6]). But there the projector was constructed by means of two mutually orthogonal vectors, one of them being timelike, the other necessarily spacelike.

Since we assumed considered variables to be second order derivable, we shall make use of an identity concerning these derivatives. A fundamental identity reads:

$$\mathcal{L}_u \mathcal{L}_\xi T_{\alpha\beta} - \mathcal{L}_\xi \mathcal{L}_u T_{\alpha\beta} = \mathcal{L}_\lambda T_{\alpha\beta}.$$

(cf Yano [3]) where $\lambda^\alpha = \mathcal{L}_u \xi^\alpha$, $T_{\alpha\beta}$ can be any tensor. Applying the above formula to $h_{\alpha\beta}$, in the case when $\nu_{\alpha\beta}$ has no components in direction u^α, one obtains, in virtue of the first relation (1.22):

$$\mathcal{L}_u \nu_{\alpha\beta} - \mathcal{L}_\xi \sigma_{\alpha\beta} = -\varphi\sigma_{\alpha\beta} \Leftrightarrow \nu_{\alpha\beta} u^\beta = 0 \tag{1.24}$$

$\sigma_{\alpha\beta}$ being the proper deformation tensor, i.e. the Lie derivative of $h_{\alpha\beta}$ with respect to u^α:

$$\sigma_{\alpha\beta} = \nabla_\alpha u_\beta + \nabla_\beta u_\alpha + u_\alpha u^\gamma \nabla_\gamma u_\beta + u_\beta u^\gamma \nabla_\gamma u_\alpha. \tag{1.25}$$

The equivalence in (1.24) follows at once from the first relation (1.22).

Multiplying (1.24) by ξ^α, we get, on account of both relations (1.22);

$$\mathscr{L}_\xi (\sigma_{\alpha\beta}\xi^\beta) = \varphi\sigma_{\alpha\beta}\xi^\beta. \tag{1.26}$$

In an appropriate local frame relation (1.26) can be expressed as an "equation of propagation" for $\sigma_{\alpha\beta}\xi^\beta$ (cf Lichnerowicz [5]). That frame is given by the comoving system of ξ^α with $\xi^\alpha(0, 0, 0, 1)$ and $\xi_\beta = g_{\beta\alpha}\xi^\alpha = g_{\beta 4}(0, 0, 0, -1)$.

Thus locally $\xi_\beta = g_{\beta 4}$. Relation (1.26) takes then the form:

$$\partial_4 \sigma_{\alpha 4} + \sigma_4^\beta \partial_\alpha g_{\beta 4} = \varphi\sigma_{\alpha 4}. \tag{1.26'}$$

Suppose that the fieled $\sigma_{\beta 4}$ was null on a spacelike hypersurface Σ, taken as an initial state surface. It results from the form of (1.26'), homogeneous in $\sigma_{\beta 4}$, that the derivatives of any order with respect to x^4 of that quantity ought to be null. We have to modify here our initial assumption of second order derivability, which was minimal, and take instead infinite derivability. The conclusion is that $\sigma_{\alpha\beta}\xi^\beta$ must remain null along the world lines of ξ^α. As it has, by definition, no components in the direction of u^α, it must become null in every 2-flat Π. Thus:

For surface forming fields u^α, ξ^α, tensor $\nu_{\alpha\beta}$ being orthogonal to an arbitrary direction in every Π (different from u^α only), and $\sigma_{\alpha\beta}$ orthogonal to ξ^α on a spacelike hypersurface Σ, both tensors have no components on surfaces S in the whole domain containing the world lines of ξ^α which intersect with Σ. In other regions of spacetime $\nu_{\alpha\beta}$ remains orthogonal to S and $\sigma_{\alpha\beta}$ identically orthogonal to world lines of u^α.

Conversely, with $\nu_{\alpha\beta}u^\beta = \sigma_{\alpha\beta}\xi^\beta = 0$, we obtain from (1.24)

$$\mathscr{L}_u (\nu_{\alpha\beta}\xi^\beta) = 0. \tag{1.27}$$

For $(\nu_{\alpha\beta}\xi^\beta)_{\Sigma'} = 0$ we have the nullity of that vector in all the region containing world lines of u^α which intersect with Σ'.

We have, from the preceding, relations (1.22) and one more:

$$\sigma_{\alpha\beta}\xi^\beta = \sigma_{\alpha\beta} h_\gamma^\beta \xi^\gamma = \sigma_{\alpha\beta}(\xi^\beta + \theta u^\beta) = 0 \tag{1.28}$$

along the ξ^α curves which intersect with Σ. By (1.25) we have then

$$\sigma_{\alpha\beta}\xi^\beta\xi^\alpha = u^\alpha \partial_\alpha(\theta^2) = 0. \tag{1.28'}$$

With (1.22) *and* (1.28) *pseudoangles between u^α and ξ^α remain constant on every 2-surface S.*

The nullity of the above quadratic form implies the locally hyperbolic character, with respect to the spacelike vector $\xi_\alpha + \vartheta u_\alpha$, of the (spacelike) proper deformation tensor $\sigma_{\alpha\beta}$.

Having written (1.28) explicity, we obtain, on account of the first relation: (1.22):

$$\mathscr{L}_u \xi_\alpha + \theta w_\alpha = 0, \quad (w_\alpha = u^\beta \nabla_\beta u_\alpha). \tag{1.29}$$

The Lie derivative of contravariant ξ^α with respect to u^α being, under our conditions, proportional to u^α; the Lie derivative of its covariant coordinates becomes proportional to the acceleration vector w_α. Thus:

$$\mathscr{L}_u \xi^\alpha \cdot \mathscr{L}_u \xi_\alpha = 0. \tag{1.30}$$

*

* *

We shall apply, to some extent, previous conclusions to relativistic magnetohydrodynamics.

Vektor field u^α is the field of four velocities of a *MHD* continuum. Maxwell's equations take the corresponding form. But except these assumptions our medium will not be specified by any energy tensor. That is perhaps not very explicit, but on the other hand, the fact that we draw some consequences of the presence of a *MHD* electromagnetic field only leaves us the possibility of applying them to any material scheme.

Electric current J^α has, in a charged medium, component parallel to u^α, the other one, due to infinite conductivity and never vanishing, being orthogonal to u^α and indeterminate. For a variable magnetic permeability μ, magnetic induction b^α ought to be written instead of h^α, without any change in the results, in what will follow.

Maxwell's equations are:

$$\nabla_\alpha (u^\alpha h^\beta - u^\beta h^\alpha) = 0 \tag{2.1a}$$

$$\frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} \nabla_\beta (h_\gamma u_\delta - h_\delta u_\gamma) = J^\alpha. \tag{2.1b}$$

Equations (2.1a) have, as a first consequence, that stream lines u^α and magnetic field lines h^α form 2-surfaces S, as can be verified at once. We shall restrict the congruence ξ^α, demanding that its families be contained in S, forming thus the same surfaces with u^α or h^α. Then, denoting by h the intensity of h^α, we shall write the first group of Maxwell's equations as:

$$\nabla_\alpha \left[\frac{hu^\alpha}{\sqrt{\vartheta^2-1}} (\xi^\beta + \vartheta u^\beta) - \frac{hu^\beta}{\sqrt{\vartheta^2-1}} (\xi^\alpha + \vartheta u^\alpha) \right] = 0, \tag{2.1a'}$$

or after developing

$$\frac{h}{\sqrt{\vartheta^2-1}} \mathscr{L}_\xi u^\beta = \nabla_\alpha \left(\frac{hu^\alpha}{\sqrt{\vartheta^2-1}} \right) \xi^\beta - \nabla_\alpha \left(\frac{h\xi^\alpha}{\sqrt{\vartheta^2-1}} \right) u^\beta. \tag{2.1a''}$$

Multiplying this expression by u_β we obtain:

$$\frac{h}{\sqrt{\vartheta^2-1}} (\nabla_\alpha \xi^\alpha + u^\alpha u^\beta \nabla_\alpha \xi_\beta) + \xi^\alpha \partial_\alpha \left(\frac{h}{\sqrt{\vartheta^2-1}} \right) \nabla_\beta u^\beta + \vartheta u^\beta \partial_\beta \left(\frac{h}{\sqrt{\vartheta^2-1}} \right) = 0. \tag{2.2}$$

Using definition formulae (1.1) and (1.26), we write this as:

$$\nu^\alpha_\alpha + \vartheta \sigma^\alpha_\alpha - h^\alpha \partial_\alpha \ln \left(\frac{\sqrt{\vartheta^2-1}}{h} \right) = 0. \tag{2.2'}$$

Devoloping this relation we can verify, in the case of proper dilatation ($\sigma^\alpha_\alpha > 0$) and of magnetic field with non increasing strength along its lines of force, that the relative expansion ν^α_α increases with the absolute value of ϑ along h^α, being then a dilatation, with $\nu^\alpha_\alpha > \sigma^\alpha_\alpha$. For proper contraction ($\sigma^\alpha_\alpha < 0$) conclusions are symmetrically opposite.

For $h^{-1}\sqrt{\vartheta^2-1}$ constant along magnetic field lines, which is, in particular, the case of a magnetic field having constant strength and pseudoangles along every line of force, ν^α_α and σ^α_α are of the same sign, with $|\nu^\alpha_\alpha| \geq |\sigma^\alpha_\alpha|$. Both vanish simultaneously.

Equations (2.1″) can be written, with the help of (2.2′) in an alternative form:

$$\text{(2.3)} \qquad \mathscr{L}_\xi u^\beta = \left(\frac{h}{\sqrt{\vartheta^2-1}}\right)^{-1}\left[\sigma^\alpha_\alpha + u^\alpha \partial_\alpha \ln\left(\frac{h}{\sqrt{\vartheta^2-1}}\right)\right] h^\beta + u^\alpha u^\gamma \nabla_\alpha \xi_\gamma \cdot u^\beta.$$

We obtain from these relations, using the first of (1.15), that for an incompressible fluid (in the kinematical sense $\sigma^\alpha_\alpha = 0$) with a constant ratio $h/\sqrt{\vartheta^2-1}$ along each streamline (proper time independence), relative deformation tensor $\nu_{\alpha\beta}$ has no components tangent to these lines.

Combining with previous results we are led to the fact that *for a MHD fluid having constant ratio $h/\sqrt{\vartheta^2-1}$ in each 2-surface S, generated by stream and magnetic field lines, and kinematically incompressible either in the proper or in the relative sense, the other kind of incompressibility also holds, and $\nu_{\alpha\beta}$ is orthogonal to its stream lines.*

In virtue of (1.26) or (1.27), we obtain, in addition to preceding conditions, that *when either the proper or the relative deformation tensor is orthogonal to the magnetic field, with the same inital condition on the other one, given on a spacelike hypersurface Σ or Σ', then both are orthogonal to the stream and the magnetic field lines along world lines ξ^α, resp. u^α, which intersect with Σ resp Σ'.*

A case when $\sigma_{\alpha\beta}$ is orthogonal to the magnetic field was obtained in [10], for $\mathscr{L}_u h_\alpha = 0$, $\nabla_\alpha u^\alpha = 0$. Expansion-free *MHD* was considered in (Bray [12]).

In order to complete the preceding we shall consider relative vorticity tensor $\Omega_{\alpha\beta}$, already defined in [11]. Although this tensor is purely kinematical, like $\nu_{\alpha\beta}$, we introduce it in this section in connection with *MHD*, in order to obtain some further relations. Relative vorticity tensor reads:

$$\text{(2.4)} \qquad \Omega_{\alpha\beta} = \nabla_\alpha \xi_\beta - \nabla_\beta \xi_\alpha + u_\alpha \xi^\gamma \nabla_\gamma u_\beta - u_\alpha u^\gamma \nabla_\beta \xi_\gamma + u_\beta u^\gamma \nabla_\alpha \xi_\gamma - u_\beta \xi^\gamma \nabla_\gamma u_\alpha.$$

By (1.1) we have:

$$(\nu_{\alpha\beta} + \Omega_{\alpha\beta}) u^\beta = 0.$$

Then

$$\text{(2.5)} \qquad \nu_{\alpha\beta} u^\beta = 0 \Leftrightarrow \Omega_{\alpha\beta} u^\beta = 0.$$

The conclusion is: *for a fluid having $h/\sqrt{\vartheta^2-1}$ constant in S, with vanishing σ^α_α or ν^α_α, relative vorticity tensor is always orthogonal to stream lines as a consequence of the orthogonality of $\nu_{\alpha\beta}$.*

The preceding can be expressed by means of a vorticity vector ψ^α, defined as (cf [1], [11]):

$$\psi^\alpha = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} u_\beta \Omega_{\gamma\delta}. \tag{2.6}$$

In considered case, when (2.5) holds, $\Omega_{\alpha\beta}$ reduces to its spacelike projection $\Phi_{\alpha\beta}$:

$$\Omega_{\alpha\beta} = \Omega_{\gamma\delta} h^\gamma_\alpha h^\delta_\beta = \Phi_{\alpha\beta}. \tag{2.7}$$

The absence of vorticity, defined by the vanishing of ψ^α, involves then the nullity of $\Phi_{\alpha\beta}$. From (2.6):

$$\psi^\alpha = 0 \Rightarrow u_\beta \Phi_{\gamma\delta} + u_\gamma \Phi_{\delta\beta} + u_\delta \Phi_{\beta\gamma} = 0.$$

Multiplying by u^β, on account of the spacelike character of $\Phi_{\alpha\beta}$ we have

$$\psi^\alpha = 0 \Leftrightarrow \Phi_{\beta\gamma} = 0.$$

The nullity of ψ^α, when added to preceding conditions, leads to the vanishing of the relative vorticity tensor $\Omega_{\alpha\beta}$.

We shall obtain, using (2.1b), a relation between the electric current vector $\mathcal{J}^\beta$ and the spacelike vorticity vector ψ^β we introduced.

When multiplying (2.1b) by ξ_β we obtain:

$$-\frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} u_\gamma h_\delta (\nabla_\alpha \xi_\beta - \nabla_\beta \xi_\alpha) = \xi_\beta \mathcal{J}^\beta.$$

The form of the left hand side is due to the fact that ξ^α can be locally expressed as a linear combination of u^α and h^α. Therefrom, by definition formula (2.6):

$$h_\alpha \psi^\alpha = \xi_\alpha \mathcal{J}^\alpha. \tag{2.8}$$

For a vorticity vector field orthogonal to S, the electric current is orthogonal to ξ_α and vice versa.

This result is analogous to the one obtained in (Yodzis [9]) for proper vorticity vector ω^α and four velocity u^α instead of ψ^α and ξ^α.

*

* *

We shall consider, finally, some relations between scalar invariants of considered tensors.

Putting $\nu^2 \equiv \nu_{\alpha\beta} \nu^{\alpha\beta}$ and $\Omega^2 \equiv \Omega_{\alpha\beta} \Omega^{\alpha\beta}$ we form the differences:

$$\nu^2 - \Omega^2 = 4 [\nabla_\alpha \xi^\beta \cdot \nabla_\beta \xi^\alpha + 2 u^\alpha \nabla_\alpha \xi^\beta u_\gamma \nabla_\beta \xi^\gamma + (u^\alpha u^\beta \nabla_\alpha \xi_\beta)^2]. \tag{3.1}$$

Then, having in mind the orthogonality to u^α of $\tau_{\alpha\beta}$ and $\Phi_{\alpha\beta}$:

$$\begin{aligned} \tau^2 - \Phi^2 &= 4 \nabla_\beta \xi^\alpha (\nabla_\alpha \xi^\beta + u_\alpha u^\gamma \nabla_\gamma \xi^\beta + u^\beta u^\gamma \nabla_\alpha \xi_\gamma + u_\alpha u^\beta u^\gamma u^\delta \nabla_\gamma \xi_\delta) = \\ &= 4 [\nabla_\alpha \xi^\beta \cdot \nabla_\beta \xi^\alpha + 2 u^\alpha \nabla_\alpha \xi^\beta \cdot u_\gamma \nabla_\beta \xi^\gamma + (u^\alpha u^\beta \nabla_\alpha \xi_\beta)^2]. \end{aligned} \tag{3.2}$$

Hence

$$\nu^2 - \Omega^2 = \tau^2 - \Phi^2. \tag{3.3}$$

The difference between the first quadratic invariants of $\nu_{\alpha\beta}$ *and* $\Omega_{\alpha\beta}$ *is equal to the difference between the invariants of their projections* $\tau_{\alpha\beta}$ *and* $\Phi_{\alpha\beta}$.

In order to examine a particular case we write

$$\nu^2 - \Omega^2 = 4\lambda_{\alpha\beta}\lambda^{\alpha\beta} \quad (\lambda_{\alpha\beta} = \nabla_\alpha \xi_\beta + u_\beta u^\gamma \nabla_\alpha \xi_\gamma). \tag{3.1'}$$

The non symmetric tensor $\lambda_{\alpha\beta}$ satisfies the condition

$$\lambda_{\alpha\beta} u^\beta = 0. \tag{3.2}$$

Let us assume that it is symmetric. It becomes then spacelike, with real spacelike axes. Thus the quadratic form (3.1′) is definite, and we have:

$$\lambda_{[\alpha\beta]} = 0 \Rightarrow \nu^2 - \Omega^2 \geqslant 0. \tag{3.3}$$

Therefrom $\nu^2 = \Omega^2$ implies the vanishing of $\lambda_{\alpha\beta}$. Multiplying that tensor by ξ^α we obtain that any convariant derivative of ξ_β is then equal to zero:

$$\lambda_{\alpha\beta} = 0 \Rightarrow \vartheta u^\gamma \nabla_\alpha \xi_\gamma = 0 \Rightarrow \nabla_\alpha \xi_\beta = 0. \tag{3.4}$$

The symmetry of $\lambda_{\alpha\beta}$ and the equality of the first quadratic invariant ν^2 and Ω^2 have the consequence that ξ_β becomes a covariant constant, being either constant in minkowskian spacetime or making the metric static in the general relativistic case. It can be verified at once that $\nu_{\alpha\beta}$ becomes then twodimensional and timelike, with purely imaginary eigenvalues, being not thus a normal tensor.

REFERENCES

[1] A. Lichnerowicz, *Théories relativistes de la gravitation et de l'électromagnétisme*, Masson (1955).

[2] J. Synge, *Relativity, the Special Theory*, North-Holland (1956).

[3] K. Yano, *The Theory of Lie Derivatives*, North-Holland (1957).

[4] J. Ehlers, Akad. Wiss. Abh. Math. Naturwiss., № 11 (1961).

[5] A. Lichnerowicz, *Relativistic Fluid Mechanics and Magnetohydrodynamics*, Benjamin (1967).

[6] Ph. Greenberg, Jr. Math. Anal. and Appl., 30, p 128—143 (1970).

[7] Pham Mau Quan, C. I. M. E. 1970 (Relativistic Hydro and Magnetohydrodynamics), Edizioni Cremonese (1971).

[8] R. Grassini, Richerche Matematiche, Napoli, XX, p 241—252 (1971).

[9] P. Yodzis, Phys. Rev. D, Vol. 3, p 2491 (1971).

[10] I. Lukačević, Theoretical and Applied Mechanics, 1, p 23—32, Belgrade (1975).

[11] I. Lukačević, Publ. Inst. Math. 19 (33), p 101—110, Belgrade (1975).

[12] M. Bray, C. R. Acad. Sc. A-B 282, A 127—130, Paris (1976).

PMF, Univerzitet
Studentski trg 16
11000 Beograd Yugoslavia