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AN INCLUSION THEOREM IN THE GENERAL THEORY OF MATRIX CONVERGENCE METHODS

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Here the following notations are used:

s — the set of all real (complex) sequences $x=(x_k)$; m, c and c_0 — the sets of the bounded, convergent and null-convergent sequences, respectively; m_0 — the set of all zero-one sequences; $e=(1, 1, 1, \ldots)$, $e_0=(1, 0, 0, \ldots)$, $e_1=(0, 1, 0, 0, \ldots)$, $e_2=(0, 0, 1, 0, 0, \ldots)$, $\ldots$; $L(X)$ — the linear manifold determined by $X \subseteq s$; $\overline{X}$ — the closure in the space m of the set $X \subseteq m$;

$A=(a_{nk})$ — an infinite matrix and the corresponding convergence method, too, A^c — the convergence domain of a method A, i. e.

$$A^c=\left\{(x_k) \mid (\forall n)\left(\sum_{k=0}^{\infty} a_{nk}\, x_k \text{ converges}\right) \wedge \left(\exists \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}\, x_k\right)\right\}.$$

As it is known, the bounded convergence domain $A^c \cap m$ of a method A is the topic in numerous research works in the convergence theory. Our theorem establishes an important characteristic of the subsets of $A^c \cap m$.

Theorem. *The condition*

$$\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty \tag{1}$$

implies

$$(\forall X \subseteq s)\ (\overline{X \cap A^c \cap m} \subseteq A^c).$$

Proof. Like L. Włodarski ([1], Th. XIX), one can prove that the condition (1) implies that the set $A^c \cap m$ is closed in the space m. Therefore we have

$$\overline{X \cap A^c \cap m} \subseteq A^c \cap m,$$

i. e.

$$\overline{X \cap A^c \cap m} \subseteq A^c.$$

Remark 1. The proved theorem is equivalent to the following assertion:

The conditions (1) *and* $X \subseteq A^c \cap m$ *imply* $\overline{X} \subseteq A^c$.

Corollary. *The conditions* (1) *and* $X \subseteq A^c \cap m$ *imply* $\overline{L(X)} \subseteq A^c$. (This is a consequence of the implication: $X \subseteq A^c \cap m \Rightarrow L(X) \subseteq A^c \cap m$.)

Now we have an operative assertion:

Let Y be a subspace of the space m and let A^c include a fundamental set X of the space Y. Then the condition (1) *implies that A^c includes the set Y, too.*

Example 1. The last assertion generalizes the two well-known results:

The conditions (1) and

$$(\forall k)(\exists \lim_{n\to\infty} a_{nk}) \tag{2}$$

imply $c_0 \subseteq A^c$;

The conditions (1), (2) and

$$\exists \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} \tag{3}$$

imply $c \subseteq A^c$.

Indeed, in the first case, the condition (2) means

$$X = \{e_0, e_1, e_2, \ldots\} \subseteq A^c.$$

Since this set is fundamental in the space c_0, we have $c_0 = \overline{L(X)} \subseteq A^c$.

In the second case we need the set $X = \{e, e_0, e_1, e_2, \ldots\}$ only.

Remark 2. When already $X \subseteq A^c$ implies the condition (1), then the following more simple assertion is valid: *Let Y be a subspace of m and let A^c include a fundamental set X of the space Y. Then A^c includes Y, too.*

Example 2. The remark 2 generalizes the following well-known result: $m_0 \subseteq A^c \Rightarrow m \subseteq A^c$.

To establish this fact, observe that m_0 is a fundamental set in the space m, i. e. $\overline{L(m_0)} = m$. Consequently, we need only to prove that $m_0 \subseteq A^c$ implies the condition (1).

At first: $m_0 \subseteq A^c \Rightarrow (\forall n)\left(\sum_{k=0}^{\infty} |a_{nk}| < \infty\right)$. To prove this, suppose

$$(\exists n_0)\left(\sum_{k=0}^{\infty} |a_{n_0 k}| = \infty\right)$$

and set

$$x_k = \operatorname{sign} a_{n_0 k} \qquad (k = 0, 1, 2, \ldots).$$

Then

$$\sum_{k=0}^{\infty} a_{n_0 k} x_k = \sum_{k=0}^{\infty} |a_{n_0 k}| = \infty.$$

Therefore $x=(x_k)\notin A^c$. Using the representation $x=y-z$, $y=(y_k)\in m_0$, $z=(z_k)\in m_0$, we have $y\notin A^c$ or $z\notin A^c$, which contradicts to the supposition $m_0\subseteq A^c$.

Now suppose $\sup_n \sum_{k=0}^{\infty} |a_{nk}|=\infty$. Without a restriction, we can assume

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{nk}|=\infty.$$

Because $e_n\in m_0$ $(n=0, 1, 2, \ldots)$, we have

$$(\forall k)\,(\exists \lim_{n\to\infty} a_{nk})$$

and therefore

$$(\forall k)\left(\sum_{\nu=0}^{k} |a_{n\nu}|=o\left(\sum_{\nu=k+1}^{\infty} |a_{n\nu}|\right),\ n\to\infty\right).$$

By Lemma II.1 from Peyerimhoff [2], p. 10, there exist two sequences $0<\nu_1<\nu_2<\nu_3<\cdots$ and $0<n_1<n_2<n_3<\cdots$ such that

$$(\forall i=1, 2, 3, \ldots)\left(\sum_{\nu=\nu_i}^{\nu_{i+1}-1} |a_{n_i,\nu}|\geqq\frac{i-1}{i}\sum_{\nu=0}^{\infty} |a_{n_i,\nu}|\right).$$

Set

$$x_\nu=\begin{cases}\operatorname{sign} a_{n_i,\nu}, & \nu_i\leqq\nu<\nu_{i+1} \qquad (i=2, 3, 4, \ldots),\\ 0, & \text{otherwise.}\end{cases}$$

Then

$$\left|\sum_{\nu=0}^{\infty} a_{n_i,\nu}x_\nu\right|\geqq\sum_{\nu=\nu_i}^{\nu_{i+1}-1} |a_{n_i,\nu}|-\sum_{\nu=0}^{\nu_i-1} |a_{n_i,\nu}|-\sum_{\nu=\nu_{i+1}}^{\infty} |a_{n_i,\nu}|=$$

$$=2\sum_{\nu=\nu_i}^{\nu_{i+1}-1} |a_{n_i,\nu}|-\sum_{\nu=0}^{\infty} |a_{n_i,\nu}|\geqq\left(2\cdot\frac{i-1}{i}-1\right)\sum_{\nu=0}^{\infty} |a_{n_i,\nu}|=$$

$$=\frac{i-2}{i}\sum_{\nu=0}^{\infty} |a_{n_i,\nu}|\to\infty \qquad (i\to\infty).$$

Therefrom $x=(x_\nu)\notin A^c$. Consequently (as before), there exists $y=(y_\nu)\in m_0$ such that $y\notin A^c$. This contradicts to the supposition $m_0\subseteq A^c$, again.

So the implication

$$m_0\subseteq A^c \Rightarrow \text{the condition (1)}$$

is true.

R e m a r k 3. It can easily be verified that all results of this note remain true for the general continuous function convergence methods (X, F, x') (see: [3], Definition 3.1; [4], Definition 2.1; [5], Corollary 3), whereby the condition (1) must be changed by

$$\sup_{x\in X} \sum_{j=0}^{\infty} |f_j(x)|<\infty.$$

(Consequently, these results are valid for continuous function convergence methods of Włodarski [6] and Orlicz [7], too.) In connection with the analogy of Example 2, see also [8], § 3.

REFERENCES

[1] W ł o d a r s k i, L.: *Sur les méthodes continues de limitation* (II), Stud. Math. XIV (1954), pp. 188—199.

[2] P e y e r i m h o f f, A.: *Lectures on Summability*, Berlin-Heidelberg-New York 1969.

[3] Л а з и ч, М. Г.: *О функциональных преобразованиях последовательностей* (I), Publ. Inst. Math. 14 (28) (1972), pp. 83—95.

[4] L a z i ć, M. G.: *On some classes of linear function transformations of sequences* (I), Publ. Inst. Math. 18 (32) (1975), pp. 117—124.

[5] L a z i ć, M. G.: *On some classes of linear function transformations of sequences* (III), Mat. vesnik 12 (27) (1975), pp. 71—80.

[6] W ł o d a r s k i, L.: *Sur les méthodes continues de limitation* (I), Stud. Math. XIV, pp. 161—187.

[7] O r l i c z, W.: *On the summability of bounded sequences by continuous methods*, Bull. de l'Acad. Polonaise des Sci., Sér. des Sci. math., astr. et phys., Vol. VI, No. 9, 1958, pp. 549—556.

[8] W ł o d a r s k i, L.: *On some strong continuous summability methods*, Proc. London Math. Soc., III, Ser. 13, pp. 273—289.

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