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RAMIFIED SETS OR PSEUDOTREES

Kurepa Đuro

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0. Ramified sets [r. s.] or **pseudotrees** were introduced in KUREPA **[1935]** p. 69, 127 as ordered sets $(R, \leq)$ such that for every $r \in R$ the set $R(\cdot, r)$ $(=(\cdot, r)_R := \{x \mid x \in R \wedge x < r\}$ is a chain.

0:1. The most important case of r. s. are *ramified tables* or *trees* satisfying the condition that every subchain is a well-ordered set. The study of ramified sets is a natural extension of the study of trees, and an easy one, if one assumes the general continuum hypothesis.

0:2. In the present paper we exhibit a method of formation of pseudotrees (theorem 2:4), natural total ordering of pseudotrees (§ 5), showing how independence (antichainness) in $(R, \leq)$ and disjointness of intervals in $(R, \prec)$ are connected.

In § 6 we introduce the notion of *pseudo convexity* and the one of *pseudo-intervals* of ordered sets.

In § 7 we prove that the question of D-reflexivity of pseudotrees is equivalent to the problem of the D-reflexivity of trees and consequently is independent of usual axioms in Set Theory (as was announced in KUREPA [1935, 1936] and proved in BUKOVSKI [1966], JECH [1967]. SOLOVAY-TENNENBAUM [1971]). In § 8 a tree-chain alternative is proved for ramified sets (Th. 8:1).

0:3. In our thesis we restricted us to trees because for the purpose we wanted both considerations are equivalent. In particular, the chain-tree alternative for pseudotrees (v. № 8) and the proposition 7:3 concerning the D-reflexivness of pseudotrees we found already in 1935.

1. Cofinality. Coinitiality of ordered sets. For any oriented pair (A, B) of subsets of an ordered set $(E, \leq)$ we define following sets:

1:1. $(A,.)_B^{\cup} := \bigcup_{a \in A} (a,.)_{(B, \leq)}, [A,.)_B^{\cup} := \bigcup_{a \in A} [a,.)_{(B, \leq)}$

1:2. $(A,.)_B^{\cap} := \bigcap_{a \in A} (a,.)_B, [A,.)_B^{\cap} := \bigcap_{a \in A} [a,.)_{(B_1 \leq)}$ and dually:

1:3. $(.,A)_B^{\cup} := \bigcup_{a \in A} (.,a)_B, (.,A]_B^{\cup} := \bigcup_{a \in A} (.,a]_{(B_1, \leq)}$

1:4. $(.,A)_B^{\cap} := \bigcap_{a \in A} (.,a)_B, (.,A]_B^{\cap} := \bigcap_{a} (.,a]_{(B_1, \leq)}$.

1:5. Cofinality (Definition) A is cofinal to B in $(E, \leq)$, symbolically A cof B in $(E, \leq)$ $\Leftrightarrow$ $(.,A]^{\cup}_{(E\leq)} = (.,B]^{\cup}_{(E\leq)}$. This means that for every $a \in A$ there is some $b, b = b(a)$ of B such that $a \leq b(a)$, and vice versa: for every $b \in B$ there is some $a = a(b)$ of A such that $b \leq a(b)$.

1:6. Lemma. *The relation of cofinality in a given set $(E, \leq)$ is an equivalence relation.*

1:7. Dually, one defines *coinitiality* of sets: coinitiality in $(E, \leq)$: $\Leftrightarrow$ cofinality in $(E, \geq)$.

2. Formation of ramified sets.

2:1. **Set $r(E, S)$.** Let (E, S) be any oriented pair of an ordered set $(E, \leq)$ and any set S; for any chain $L \subset (E, \leq)$ we consider the set S^L of all single-valued mappings of L into S; let.

2:2. $L(E, \leq)$ be the system of all subchains of $(E, \leq)$; we define the set

2:3. $r = r(E, S) := U_X S^X$, $(X \in L(E, \leq))$ ordered by $\sqsubseteq|$, where for $(x, x') \in r \times r$ we denote by $x \sqsubseteq| x'$ that $x = L | S$, $x' = L' | S$ are such that:

1) L, L' are chains in $(E, \leq)$

2) L is an initial section of L' symbolically $L \subset_i L'$;

3) $x | L = x' | L$.

2.4. Theorem. *The set r is ramified. Every ramified set $(R, \leq)$ is isomorph to a subset of ramified set of the form* (2:3).

Proof. Let $\{f, g, h\} \subset (2:3)$ and $\{f, g\} \sqsubseteq| h$; then

$$f: L_1 \to S, \quad g: L_2 \to S, \quad h: L_3 \to S$$

where $L_1, L_2, L_3 \in L(E, \leq)$; moreover since $\{f, g\} \sqsubseteq| h$ one has $f \sqsubseteq| f = h | L_1$, $g = h | L_2$; L_1, L_2 being initial sections of the chain L_3 one has $L_1 \subset_i L_2$ — and therefore $f \sqsubseteq| g$ — or $L_2 \subset_i L_1$ and consequently $g \sqsubseteq| f$. Thus $\{f, g, h\}$ is a subchain of (2:3): the set (2:3) is ramified.

Let us prove that every ramified set $(R, \leq)$ is similar to a subset of sets of the form (2:3). It suffices to consider the set R of identity mappings of all the lower cones $(\cdot, x]_{(R, \leq)}$; the mapping $x \in R \to j | (\cdot, x]_{(R, \leq)}$ is an imbedding of $(R, \leq)$ into $r((R, \leq), R)$. Thus the theorem 2:4 is proved.

2:5. **A Problem of economisation.** In connexion with the theorem 2:4 the following problem arises: *given $(R, \leq)$, find an imbedding into $r(E, \leq), S)$ with arguments $(E, \leq)$, S that are as simple as possible.*

2:5:1. A first reduction is that in the theorem 2:4 one could assume that $(E, \leq)$ be a chain.

This reduction is an obvious consequence of the fact that the order $(R, \leq)$ allows a total extension.

2:5:2. A second reduction consists to assume that S be a set of ordinal numbers and that $|S| \leq \check{s}(R, \leq)$.

As a matter of fact, let $(T, \leq)$ be a subtree of $(R, \leq)$ that is coinitial to $(R, \leq)$. Then we have the height $\gamma(T, \leq)$ and the disjoint partition $T = \cup_k R_k T (k < \gamma T)$; for every row $R_k T$ we consider the partition

2:5:3. $N(R_k T) := \{|x|_{(T,\leq)} \therefore x \in R_k T\}$, where $|x|_{(T,\leq)} :=$

$$\{y \therefore y \in T, (\cdot, x)_{(T,\leq)} = (\cdot, y)_{(T,\leq)}\}$$

and a normal well-ordering

2:5:4. $a_{k0}, a_{k1}, \ldots, a_{kj}, \ldots$ $(j < \tau_k)$ of $N(R_k T)$ as well as a normal well-ordering

2:5:5. a_{kjs} $(s < \gamma_{kj})$ for every $a_{kj} \in N(R_k T)$; γ_{kj} is the ordinal rank of the well-ordering of a_{kj}; therefore, $\gamma_{kj} < \omega_0 V \gamma_{kj} = \omega_\xi$, for some $\xi = \xi(k, j)$. To start with, let us remark that $N(R_0 T) = \{R_0 T\}$ and that a_{00s} $(s \subseteq \gamma_{00})$ are all points of a_{00} $(= R_0 T)$. We have the point $a_{000} \in a_{00}$ and the corresponding left-interval--the chain-$(\cdot, a_{00}]_{(R,\leq)} := L$.

We have to define the function $f_0 | (\cdot, R_0 T]^{\cup}_{(R,\leq)}$. We start by setting $f_0 = 0$ on the chain L. For every $0 < s < \gamma_{00}$ let us assume that the mapping f_0 be defined on the set

$$(\cdot, a_{00s}] \setminus \cup_{i<s} (\cdot, J_{00i}]_{(R,\leq)} \text{ to be } f_0 = s.$$

By induction, the function $f_0 | (0,0]_{(R,\leq)}$ is defined where, for abbreviation $(\cdot, k] := \cup_{i \leq k} (\cdot, R_k R]^{\cup}_{(R,\leq)}$. Obviously, f_0 is an isomorphism.

Assume now $0 < k < \gamma(T)$ and that for every $j < k$ a mapping $f_j | (\cdot, j]$ is defined such that $0 \leqslant i < j$ implies that f_i is a subfunction of f_j and that the mapping

$$(\cdot, x]_{(R,\leq)} \to f_j (\cdot, x]_{(R,\leq)} (x \in (, j])$$

is an isomorphism. We are going to define a mapping $f_k | (\cdot, k]$ in the following way: $f_k(\cdot, j] := f_j | (\cdot, j]$ $(j < k)$. $f_k = 0$ on the set $(\cdot, a_{k00}] \setminus (\cdot, k)$ $f_k = s$ on the set $(\cdot, a_{kos}] \setminus (0, k) \setminus \cup_{j<s} (\cdot, a_{koj}]$ for every $s < \gamma_{k0}$ $f_k := s$ on the set $(\cdot, a_{kjs}]_{(R,\leq)} \setminus (\cdot, k) \setminus \bigcup_{i<k} (\cdot, a_{ki}]_{(R,\leq)} \setminus \cup_{j<s} (\cdot, a_{kij}]$.

2:5:6. It is easy to verify that f_k is an isomorphic imbedding of $(\cdot, k]$. At first the mapping f_k is single-valued. The mapping f_k is stricly increasing: if $x < x'$ in $(\cdot, k]_{(R,\leq)}$, then $f_k x \dashv f_k$. As a matter of fact, by induction hypothesis, if $y < k$ the mapping $f_j | (\cdot, j]_{(R,\leq)}$ is an isomorphism; thus if $\{x, x')_{\neq} \subset (\cdot, j]_{(R,\leq)}$ for some $j < k$, then $f_k : \{x, x'\}$ is an isomorphism; therefore let us consider the case that $\{x, x'\} \subset (\cdot, j]$ for no $j < k$; this means that the ordinal k is non limit, and > 0; we have the following two cases:

$$\{x, x'\} \cap (\cdot, k-1] = \varnothing \quad \text{or} \quad |\{x, x'\} \cap (\cdot, k-1]| = 1.$$

CASE: $\{x, x'\} \cap (\cdot, k-1] = \varnothing$.

Let $t(x)$ be the point of $R_{k-1} T$ such that $t(x) \leq x$; let a_{kj} be the first member in the well-ordering (2:5:4) of $N(R_k T)$ such that $x \leq a_{kj}$ and let a_{kjs} be the first member in the well-ordering (2:5:5) of a_{kj} such that $x \leq a_{kjs}$. Analogously, one has the point $t(x')$, the set a_{kj}, and the point $a_{kj's}$, related to x'. The chains $[t(x), a_{kjs}]_{(R,\leq)}$, $[t(x') a_{kj's'}]_{(R,\leq)}$ are completely determined, Now, if $x \| x'$, then the chains $(\cdot, x]_{(R,\leq)}$, $(\cdot, x']_{(R,\leq)}$ are $\subset$-incomparable, thus $(f(x), f(x'))$ are $\sqsubseteq|$-incomparable. If $x < x'$, then $t(x) = t(x')$; if moreover $j' < j$, then the definition of $f_k(x')$ was given before that of $f_k x$, i. e. $f_k(\cdot, x']$

implied the determination of fx and obviously $fx \equiv| \neq fx'$. The same holds if $j'=j'$, $s'<s$. If $j'>j$, then the definition of $f_k x$ preceded that one of $f_k x'$, therefore again $fx \equiv| \neq fx'$. Consequently, for every $k<\gamma T$, the isomorphism of f_k is proved. Similarly, if $x'<x$, then $fx' \equiv| \neq fx$.

2:5:7. This being so, let us define the mapping $f|(R, \leq)$ by the equalities $f|(\cdot, R_k T]^{\cup}_{(R, \leq)} := f_k$ for every $k<\gamma T$. The isomorphic imbedding f is implied by the same property of f_k for every $k \subset \gamma T$ — the theorem 2:4 is completely proved.

2:6. Remark. Every fx in the preceding considerations could be meant of as a word, the characters of which are ordinals. Consequently, the image $f(R, \leq)$ of $(R, \leq)$ could be considered as a certain vocabulary.

3. A triangular lemma on ramified sets.

3:1. *Definition of* E_{ab}. For any $(a, b) \in (E, \leq)^2$ let

$$E_{ab} := \{x \mid x \in E,\ x \leq \{a, b\}\};\ \text{i. e.}$$

$E_{ab} := (\cdot, a]_{(E, \leq)} \cap (\cdot, b]_{(E, \leq)}$. Then we have the following lemma showing that in the case of ramified sets R the mapping $R_{ab}: R \times R \to PR$ behaves like a non numerical distance (cf. KUREPA Đ. [1956]. p. 108).

3:2. Lemma. *If R is any ramified set, then any* $\{a, b, c\} \subset R$ satisfies

3:3. $|\{R_{ab}, R_{bc}, R_{ca}\}| \leq 2$;[1] more precisely

3:4. $(R_{ab} \in \{R_{ac} | R_{cb}\}) \vee (R_{ac} = R_{cb})$.

Proof. At first

3:5. $R_{ab} \supset R_{ac} \cap R_{cb}$ because if $x \in (3:5)_2 :=$ second part of (3:5), then $x \leqslant \{a, c, b\}$ thus $x \leqslant \{a, b\}$ and consequently $x \in (3:5)_1$. On the other hand, let us assume

3:6. $y \in (3:5)_1$ and $R_{ac} \neq R_{cb}$. We shall prove that then $y \in (3:6)_2$. Now R_{ac}, R_{cb} as initial portions of the chain $(\cdot, c]_R$ are $\subset$-comparable: one has

3:7. $R_{ac} \subset R_{cb}$ or 3:8. $R_{ac} \supset R_{cb}$.

3:9. Case (3:7); then $(3:5)_2 = R_{ac}$. For the points c, y we have either $y<c$ or $y=c$ or $y>c$ or $c \| y$.

3:9:1. $\neg(y=c)$: If $y=c$, then, since $y \leqslant a, b$ we infer that $c \leqslant a, b$ thus $R_{ac} = R\ \cdot, c] = R_{cb}$ thus $R_{ac} = R_{cb}$, contrarily to (3:6).

3:9:2. $\neg(y>c)$. If $y>c$, then $c<y \leqslant \{a, b\}$, thus $c<\{a, b\}$ and again $R_{ac} = R(\cdot, c] = R_{cb}$, consequently, $R_{ac} = R_{bc}$, contrarily to (3:6).

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[1] (Added 1977:10:14:5) While reading the manuscript of this paper, Todorčević Stevo, a student, remarked that the converse of 3:2 was holding:

3:2′. Lemma: *If an ordered set* $(E, \leq)$ *is such that* $\{a, b, c\} \subset E \Rightarrow |\{E_{ab}, E_{bc}, E_{ca}\}| \leq 2$, *then* $(E, \leq)$ *is ramified.*

Proof. In opposite case, there would be a point $a \in R$ and two incomparable points b, c of E such that $\{b, c\} \leqslant a$. Obviously, $a \notin \{b, c\}$, $E_{ab} = (\cdot, b]_{(E, \leq)}$, $E_{ac} = (\cdot, c]_{(E, \leq)}$ and $b, c \notin E_{bc}$; consequently, $b \in E_{ab} \setminus E_{ac}$, $c \in E_{ac} \setminus E_{ab}$, $b \in E_{ab} \setminus E_{bc}$, $c \in E_{ac} \setminus E_{bc}$, i. e. $|\{E_{ab}, E_{bc}, E_{ac}\}| = 3$, in contradiction with the assumption.

3 : 9 : 3. $\neg(c \| y)$. If $c \| y$ then also $a \| c$, $b \| c$. As a matter of fact, one has neither $a \leq c$ (otherwise $y \leq c$) nor $a > c$ (otherwise, c, y should be 2 incomparable predecessors of a). Consequently

3 : 9 : 4. $R_{ac} = R_{yc}$. Similarly, we prove that b, y are incomparable and that

3 : 9 : 5. $R_{bc} = R_{yc}$. From (3 : 9 : 4), (3 : 9 : 5) we would infer that $R_{ac} = R_{bc}$, contrarily to the opposite assumption (3 : 6). Consequently, since neither $y = c$ nor $y > c$ nor $y \| c$, one has $y < c$, then also $y \leqslant \{a, c\}$ (because $y \in (3:5)_1$ and thus $y \in R_{ac} = (3:5)_2$.

3 : 10. Case 3 : 8: Permuting a, b in (3 : 9) one proves again (3 : 4). The statement (3 : 4) is completely proved.

4. Ground blocks of any ordered set.

For every non empty ordered set $(M, \leq)$ we have the following system $B(M, \leq)$ of subsets of $(M, \leq)$:

4 : 1. $B(M, <) := \{[L, \cdot)_M^{\cup} := \bigcup_{l \in L} [l, \cdot)_{(M, \leq)} \mid L$ is a chain in $(M, \leqslant)$ such that $m < L$ for no $m \in M\}$. Thus in particular, if l is any member of M such that there is no $m \in M$ satisfying $m < l$, then $[l, \cdot)_{(M, \leq)} \in B(M, \leq)$. Members of $B(M, \leq)$ may be called *ground or right blocks of* $(M, \leq)$.

4 : 2. Lemma. *The family* $B(M, \leq)$ *is well determined and the union of its members equals* M.

As matter of fact, let $x \in M$; if there is some x' in M such that $x' \leq m$ and $(\cdot, x')_M = \varnothing$, then m is a member in the member $[x', \cdot)_M$ of the partition. If the preceding case does not occur, let then L be a maximal chain in $(\cdot, x]_M$; then $[L, \cdot)_M$ is a member of (4 : 1) and this member contains m as its own member.

4 : 3. Δ-*sets*. A special kind of ordered sets is this one (Δ) for which every subset $(M, \leq)$ allows a disjoint initial partition. Such is the case of ramified sets:

4 : 4. *If a set is ramified, then it belongs to the class* (Δ)[1].

4 : 5. *Right node of* $(M, \leq)$ is any most extensive right section B of M such that for every maximal chains $L, L' \subset B$ we have $(\cdot, L)^{\cap}_{(M, \leq)} = (\cdot, L')^{\cap}_{(M, \leq)}$, i. e. the predecessors of L in $(M, \leq)$ are exactly the same as the ones of L'. In particular, the set $R_0(M, \leq)$ of all points of $(M, \leq)$ without any predecessor in $(E, \leq)$ are located in a same right node of $(M, \leq)$; in general, the set $R_0(M, \leq)$ is a proper part of a left node of $(M, \leq)$.

[1] And reciprocally: *If an ordered set* $(E, \leq)$ *has the* Δ-*property*, $(E, \leq)$ *is ramified* (observed by S. Todorčević, 1977:10:14:5.).

Proof. In opposite case, there should be an ordered set $(E, \leq)$ and a point $a \in E$ with at least two incomparable predecessors b, c; then obviously $B(\{a, b, c\}) = \{\{a, b\}, \{b, c\}\}$, and the blocks $\{a, c\}$, $\{b, c\}$ are not disjoint, contrarily to the assumption that $(E, \leq)$ satisfies (Δ).

5. Natural total ordering of any ramified set $(R, \leq)$.

5:1. Theorem. *I. Let $(R, \leq)$ be any ramified set; let*

5:1:1. *$I(R, \leq) := \{R_{ab} := (\cdot, a]_{(R, \leq)} \cap (\cdot, b]_{(R, \leq)} \therefore \{a, b\} \subset R, a \| b\}$; for every $X \in I(R, \leq)$ let $\leq_X$ be any total ordering of the set X^+ of all right blocks of the set of all points r of $(R, \leq)$ such that $r \geq X$; in particular, for every antichain $\{a, b\} \subset (R, \leq)$ one has: $R_{ab} \in I(R, \leq)$, and the determined total ordering*

5:1:2 *$\leq_X$ of X^+ where $X = R_{ab}$ and the system*

5:1:3. *$N(R, \leq)$ of all induced orders $\leq_{N(a, b)}$ of $(R, \leq)$, where for any $\{x, y\} \in R^2$,*

5:1:4. *$x <_{N(a, b)} y :\Leftrightarrow x \| y \wedge x <_{N(a, b)} y$, i. e.*

$$R_{ab}(x) <_{R(a, c)} R_{ab}(y), \text{ where}$$

$x \in R_{ab}(x) \in (R_{ab})^+$, $y \in R_{ab}(y) \in (R_{ab})^+$. If for every $(x, y) \in R^2$ we define $\prec$ by

5:1:5. *$x \prec y \Leftrightarrow x \leqq y \vee x \| y \wedge x \leq_{N(a, b)} y$ then $(R, \prec)$ is a total order that extends all orderings $\leq, \leq_{N(a, b)} \in N(R, \leq)$ for every incomparable pair (a, b) of elements of $(R, \leq)$. No proper subset of $N(R, \leq)$ yields jointly with $\leq$ a total order of $(R, \leq)$ in the preceding sense (cf.* Đ. Kurepa [1935] p. 127, [1938] p. 199, [1939a] p. 70—71 and G. Hoheisel — J. Schmidt [1953].

II. For every point $t \in R$ the set

5:1:6. *$[t, \cdot)_{(R, \leq)}$ is convex in $(R, \prec)$; i. e. if the set (5:1:6) contains two points a, b, then it contains the whole corresponding interval $[a, b]_{(R, \prec)}$.*

III. *For every oriented pair (a, b) of distinct points of R such that $a \prec b$, we have the following implication:*

5:1:7. *$t \in (a, b)_{(R, \prec)} \setminus (a, b)_{(R, \leq)} \Rightarrow (5:1:8)$ where*

5:1:8. *$[t, \cdot)_{(R, \leq)} \setminus [b, \cdot)_{(R, \leq)} \subset (a, b)_{(R, \prec)}$.*

5:2. Proof of the theorem 5:1:I. Let us prove that $\prec$ is transitive:

5:2:1. $(x, y, z) \in R^3 \wedge x \prec y \prec z \Rightarrow x \prec z$.

This is trivial if both $\prec$ in $x \prec y \prec z$ mean either $\leq$ or a same $\leq_{N(a, b)}$ or if $|\{x, y, z\}| = 2$. Therefore let us assume that $|\{x, y, z\}| = 3$ and that

5:2:2. $x <_{N(x, y)} y, y <_{N(y, z)} y$.

Now, by 3:3 the set $\{R_{xy}, R_{yz}, R_{zx}\}$ has ≤ 2 members, i.e. we have the tautology

5:2:3. $R_{xy} = R_{yz} \vee R_{yz} = R_{zx} \vee R_{zx} = R_{xy}$.

5:2:4. Case $R_{xy} = R_{yz} := R_y$. One has $S_{xz} \supset R_y$ i.e. $R_{xz} = R_y$ or $R_{xz} \supset \neq R_y$. In the first case, $R_{xz} = R_{yz} = R_{zx} := S$; the relations $x \prec \neq y \prec \neq z$ imply that the points x, y, z belong to distinct blocks corresponding to S; consequently, $x <_S y <_S z$, thus also $x <_S z$.

Second case: $R_y \subset \neq R_{xz}$. Let $c \in R_{xz} \setminus R_y$. Since $\neg c \leq y$, one has $c > y$ or $c \| y$. Now, $c > y$ does not hold because $c > y$, $c < z$ would imply $y < z$, contrarily to the assumption that $y \| z$. Conseqently, $c \| y$. Since $c < z$ we have $R_{yz} = R_{yc}$, thus $R_{xy}=R_{yc}$; in other words, for the set $\{x, y, z\}$ we would be in the first case and the relations $x \prec y \prec c$ would imply $x \prec c$ what jointly with $c \prec z$ would imply $x \prec z$.

By cyclic permutations of characters x, y, z one settles as well the remaining two cases $R_{yz}=R_{zx}$, $R_{zx}=R_{xy}$.

5:2:5. The mutual pairwise independence of the relations $\leq_{N(a,b)}$ is obvious from the definition of the system (5:1:3).

5:3. Proof of theorem (5:1:II). We have $t<a, b$ and $a \neq b$; thus $a \prec b$ or $b \prec a$. Assume $a \prec b$; we have to prove that for every point $c \in R$ such that $a \prec c \prec b$ one has $t<c$. In opposite case we would have either $c \leq t$ or $c \| t$. Now, one has not $c \leq t$, because $c \leq t$ would imply also $c<a$, contrarily to the assumption $a<c$.

The case $c \| t$ is not possible neither. In order to prove it let us consider the 3-point-set $\{a, b, c\}$ and the set $\{R_{ab}, R_{bc}, R_{ca}\}$ consisting of ≤ 2 members. Thus, logically one of the following three cases (5:3:1), (5:3:2), (5:3:3) should hold:

5:3:1. $R_{ac}=R_{bc}:=R_c$. Since $R_{ab} \supset R_c$, the sets R_c, $\{t\}$ belong to the chains $(\cdot, a)_{(R,\leq)}, (\cdot, b)_{(R,\leq)}$; therefore, also $R_c \cup \{t\}$ would be a chain, thus either $t \in R_c$ (consequently $t<c$, contradicting the assumption $t \| c$) or $R_c<t$; now, the relation $R_c<t$ would imply that the points c, t would be in distinct R_c-blocks containing $\{c\}$ and $\{a, b\}$ respectively and thus either $\{a, b\} \prec c$ or $c \prec \{a, b\}$, contrarily to the assumption $a \prec b \prec c$.

5:3:2. *Case* $R_{ba}=R_{ca}:=R_a$. Since $R_{ac}=R_{tc}=R_{bc}$, the points c, t would belong to distinct R_a-blocks, one containing t (thus also a, b) the other one containing c, thus either $\{a, b\} \prec c$ or $c \prec \{a, b\}$, contrarily to $a \prec c \prec b$.

5:3:3. *Case* $R_{cb}=R_{ab}$ is impossible for a similar argument. This completes the proof of the theorem (5:1. II).

5:4. Proof of the theorem 5:1:III. Since $a \prec b$, we have $a<b$ or $a \| b$.

5:4:1. *First case*: $a<b$. Since $a \prec t \prec b$, one has logically following situations:

I: $a \leq t<b$ (impossible in virtue of $t \in (5:1:7)_1$).

II: $a \leq t \| b$; put $M:=(\cdot, t]_{(R,\leq)}$; then the points b, t belong to $\neq$ blocks $M(b)$, $M(t)$ and since $t \prec b$ one has $M(t)<_M M(b)$ and thus $[t, \cdot)_{(R,\leq)} \prec b$; consequently, (5:1:7) holds.

III: $a \| t \| b$. This case is not possible because setting $M:=(\cdot, a]_R \cap (\cdot, t]_R$, one has $M(a)=M(b)$ and consequently $t \prec \{a, b\}$ or $\{a, b\} \prec t$, contrarily to the assumption $t \in (a, b)_{(R,\prec)}$.

IV: $a \| t<b$. This case is not possible because it would mean in opposite case, that the points a, t would be incomparable predecessors of a same point b, contradicting the assumption that $(R, \leq)$ is ramified.

5:4:2. *Second case*: $a \| b$. Put $L:=(\cdot, a]_{\leq} \cap (\cdot, b]_{\leq}$. Since $a \prec t \prec b$, we have to examine the following 4 cases:

I: $a \leq t < b$ (impossible because $a \| b$).

II: $a \leq t \| b$. In this case $L(a) = L(t)$, thus $L(t) <_L L(b)$ and the relation (5:1:7) is true.

III: $a \| t < b$ (v. 5:4:1. IV).

IV: $\{a, b, t\}$ is an antichain in $(R, \leq)$.
Set: $A := (\cdot, b]_{(R, \leq)} \cap (\cdot, t]_{(R, \leq)}$, and cyclically

$$B := (\cdot, t]_{(R, \leq)} \cap (\cdot, a]_{(R, \leq)}, \; T := (\cdot, a]_{(R, \leq)} \cap (\cdot, b]_{(R, \leq)}.$$

By 3:3, $|\{A, B, T\}| \leq 2$. We have to consider the following 3 cases 5:4:3—5:4:5 exhausting all imaginable cases.

5:4:3. $A = B$. Then necessarily $B \subset T$ because $A = B$ implies

$$A = B = \{x : \cdot x \in R \wedge x \leq \{a, b, t\}\} \subset \{x : \cdot x \in R \wedge x \leq \{a, b\}\}.$$

If moreover $B = T$, then $A(a) <_A A(t) <_A A(b)$ and therefore (5:1:7) holds. Now, one has not $B \subset \neq T$ because this relation would imply $B(a) = B(b)$, thus $a <_B t \wedge b <_B t$ or $t <_B a \wedge t <_B b$; therefore $t \notin (a, b)_{(R, \prec)}$, contradicting the assumption that $a \prec t \prec b$.

5:4:4. Cyclic permutation of the case 5:4:3.

5:4:5. Cyclic substitution of the case 5:4:4.

Thence the theorem 5:1:III and the theorems 5:1:I, II, III are completely proved.

Here is a corollary to the theorem 5:1:III.

5:5. Corollary. *If* $(R, \leq)$ *is ramified, then every total natural order extension* $(R, \prec)$ *of* $(R, \leq)$ *satisfies the following condition*

$\{a, b\} \subset R, a \| b, a \prec t \Rightarrow (t, .)_{(R, \leq)} \subset (a, b)_{(R, \prec)}$.

We have also the following

5:6. Lemma. *Let* $(R \leq)$ *be a ramified set and* $R, \prec)$ *a natural total order extension of* $(R, \leq)$; *if* $\{a, b, t\} \subset R$ *and* $a \prec t \prec b$, *then the relation*

$$(5:7) \qquad t \in (a, b)_{(R, \prec)} \Rightarrow (t, \cdot)_{(R, \leq)} \setminus [b, \cdot)_{(R, \leq)} \subset (a, b)_{(R, \prec)}$$

holds if and only if for the set $M := (t, \cdot)_{(R, \leq)}$, *the element* b *belongs to the last block of* BM *in the ordering.* $\leq_M$.

Proof $\Rightarrow$: If If $M(b)$ were not the last member of $(BM, \leq_{BM})$, there would be a block $X \in BM$ such that $b \prec X$ and hence $B < X$ contrarily to the assumption.

$\Leftarrow$: If b belongs to the last block, then every member x of M belongs to some block $M(x) \prec M(b)$ and thus $x \prec b$, i. e. (5:7) is satisfied.

In connexion with considerations in 5:1 we are going to introduce some intervals and a kind of convexity in ordered sets.

6. Pseudo-convexity of ordered sets. Pseudo-intervals.

6 : 1. *Pseudo-convexity.*

6 : 1 : 1. Definition. Let $(E, \leq)$ be ordered and $F \subset E$; F is said to be convex, if $\{a<b\} \subset F \Rightarrow (a, b)_{(E, \leq)} \subset_{\rightarrow} F$.

6 : 1 : 2. Definition. F is said to be *right pseudo-convex* provided

$$\{a<b\} \subset F \Rightarrow [a, \cdot)_{(E, \leq)} \setminus [b, \cdot)_{(E, \leq)} \subset F.$$

6 : 1 : 3. Definition. F is said to be *left pseudo-convex* provided

$$\{a, b\} \subset F \Rightarrow (\cdot, b]_{(E, \leq)} \setminus (\cdot, a)_{(E, \leq)} \subset F.$$

6 : 1 : 4. F is said *pseudo-convex provided F is both left pseudo-convex and right pseudoconvex.* The empty set and every singleton are considered to be pseudo-convex.

6 : 1 : 5. For chains, convexity coincides with the pseudo-convexity.

6 : 1 : 6. Of course, every pseudo-convex subset of $(E, \leq)$ is convex (the opposite need not hold. E. g. every convex part of every maximal subchain L of $(E, \leq)$ is convex in $(E, \leq)$ but need not be pseudo-convex).

6 : 2. *Pseudo-intervals of ordered sets.*

6 : 2 : 1. Let $\{a,b\}$ be a chain in an ordered set $(E, \leq)$; assume $a \leq b$; then the open pseudo-interval $a\ b$ of $(E, \leq)$ is denoted by$\}a\ b\{$and defined by $(a\}\ b\{\ )_{(E, \leq)} := a\}\ \{b := (a, \cdot)_{(E, \leq)} \setminus [b, \cdot)_{(E, \leq)} \cup (\cdot, b)_{(E, \leq)} \setminus (\cdot, a)]_{(E, \leq)}$.

6 : 2 : 2. The closed pseudo-interval is denoted and defined in the following way:

$$(\ \}a\ b\{\ )_{(E, \leq)} := \{[a, \cdot)_{(E, \leq)} \setminus (b, \cdot)_{(E, \leq)}\} \cup \{(\cdot, b]_{(E, \leq)} \setminus (\cdot, a)_{(E, \leq)}\}.$$

6 : 2 : 3. Left open interval $a\ b$ is defined like

$$(a\}\ b\{\ )_{(E, \leq)} := (a, \cdot)_{(E, \leq)} \setminus (b, \cdot)_{(E, \leq)} \cup (\cdot\ b]_{(E, \leq)} \setminus (\cdot, a]_{(E, \leq)}.$$

6 : 2 : 4. Similarly one defines the right open interval ab denoted by $(\ \}a\{b)_{(E, \leq)}$.

6 : 3. In this terminology, we see that a subset S of $(E, \leq)$ is pseudo-convex if and only if for every subchain $\{a, b\} \subset S$ the corresponding closed pseudo-interval of $(E, \leq)$ is contained in S.

6 : 4. *The set $P_{pc}(E, \leq)$ of all pseudoconvex subsets of $(E, \leq)$.* If $(E, \prec)$ is a total order extension of $(E, \leqslant)$ one has to examine whether the corresponding families of pseudoconvex sets are $\subset$-comparable or $\subset$- incomparable, in particular, if one considers any natural total extension $(R, \prec_n)$ of any ramified set $(R, \leq)$, the problem arises whether

(6 : 4 : 1) $\qquad P_{pc}(R, \leq) \subset (R, \prec_n)$ holds.

6 : 4 : 2. Problem. *Find sufficient conditions that a given ordered set $(E, \leq)$ for at least one total extension $(E, \leq)$ satisfies $P_{pc}(E, \leq) \subset P_{pc}(E, \prec)$; $P_{pc}(E, \leq) = P_{pc}(E, \prec)$ respectively.*

6 : 4 : 3. According to ST. TODORČEVIĆ if a pseudotree $(R, \leq)$ is not the union of two mutually incomparable subsets, then the relation (6 : 4 : 1) holds for every natural total extension $\prec_n$ of $\leq$.

The converse does not hold, because e. g. for the tree $(T, \leq)=\{0, 1, 2, 3, 4, 5, 6, 7\}$, where $0<1<\{2, 3, 4, 5, 6, 7\}$, no $(T, \prec_n)$ satisfies $(6:4:1)_1 = = (6:4:1)_2$ (remark that $(T, \leq)$ is not the union of twe mutually incomparable subsets).

7. Ramified sets and cofinal trees.

7.1. Lemma. *Let R be any ramified set; then there exists a tree T such that* $R=\cup_t (\,.\,, t]_R$, $(t\in T)$.

Proof. For any subset M of R let $A(M)$ be a maximal antichain; thus $M=\cup[a]_M$, $(a\in A)$. Put $T_0 := A(R)$. $T_1 := A(R\setminus \bigcup_x (\,.\,, x]$, $(x\in T_0)$; for every ordinal $\alpha>0$ we define

$$T_\alpha := A(R\setminus \bigcup_x (\,.\,, x]\ (x\in \bigcup_{i<\alpha} T_i.$$

Setting $T=\bigcup_\alpha T_\alpha$, one sees that T is a tree $\subset R$ and that $R_i T = T_i$ and that every member $x\in R$ is contained in $(\,;\, y]_R$ for some $y\in T_i$ for some $i>\gamma T$; in other words, $(T, \leq)$ is cofinal to $(R, \leq)$ (v. 1 : 5).

7.1.1. Put $\gamma_m R := \inf_T \gamma T$, T being cofinal to $(R\leq)$.

7.1.2. **Problem.** *Is the ordinal* $\gamma_m R$ *an initial number?*

7.2. Lemma *Let R be a ramified set and T a tree cofinal to R. Every maximal antichain of* $(T, \leq)$ *is a maximal antichain in* $(R, \leq)$. *To every* $L\in L_M T$ ($\equiv$*the system of all maximal chains* $\subset(T, \leq)$) *corresponds one and only one maximal chain* $fL\in L_M R$*; in this way one gets the whole set* $L_M R$, i.e. $fL_M = T_M R$.

Proof: Let A be any maximal antichain in $(T, \leq)$; if there were an antichain $A' \supset\neq A$ in $(R, \leq)$, let $a'\in A'\setminus A$. Since $(T, \leq)$ is cofinal to $(R, \leq)$, there would be a $t\in T$ such that $a'\leq t$. On the other hand, A being a maximal antichain of $(T, \leq)$, there is a point $a\in A$ such that $\{a, t)$ be a chain. One proves readily that this is not possible, because either of relations $a\leq t$ or $t<a$ yields a contradiction. Now, let $x\in L_M T$ and $fx := \cup_y (\,.\,, y]_{(R,\leq)}$, $(y\in x)$; then fx is a maximal chain of $(R, \leq)$. At first, fx is obviously a chain. If fx were not a maximal chain, there would be a point $z\in R\setminus fx$ such that $fx\cup\{z\}$ be a chain; now, $x\cup\{z\}$ is also a chain. It is not possible that $z<x'$ for some $x'\in X$, because $z\notin fz$; therefore $x<z$; on the other hand, there is some member $t\in T$ such that $z\leq t$; consequently, $x<z\leq t$ i.e. $x<t\in T$, contradicting that x is a maximal chain in T.

On the other hand, let X be any maximal chain of R; put $x=X\cap T$; then x is a chain in T. Further, x is maximal. In opposite case, there would be a point $z\in T\setminus x$ such that $x\smile\{z\}$ be a chain. Now, $x<z$ because if some $x'\in x$ satisfies $z\leq x'$ then for some point $X'\in X$ one would have $x'\leq X'$ and thus $z\in X$ – contradiction.

7.3. Theorem. *In order that a ramified set R be D-reflexive, it is necessary and sufficient that every tree be D-reflexive.*

Proof. The necessity being obvious, let us prove that the condition of the theorem 7:3 is sufficient.

7.3.1. We can assume that every subtree of $(R, \leq)$ is $<|R|$; in particular that every chain as well as every antichain of $(R, \leq)$ is $<|R|$.

7.3.2. Let $S_0 := \{x \mid x \in R, |R[x]| < |R|\}$.

7.3.3. First case; $|S_0| = |R|$. Let A be an antichain of S_0 such that $S_0 = \bigcup_a S_0[a]$, $(a \in A)$. If the set

7.3.4. $[A, .)_R^{\cup} := \bigcup_a [a, .)_{S0}$ has $|R|$ points, then it is easy to get a D-subset of $|R|$ points. Therefore let assume that the set (7:3:4) has $<|R|$ points. Then the set $(\,.\,, A]_{(R<)}^{\cup} := \bigcup_{a \in A} (\,.\,, a]_R$ has $|R|$ points. Each summand being a chain thus $<|R|$ there is a subset $A_0 \subset A$ such that

7.3.5. $\sup_x (\cdot, a]_A = |R|$ where $x \in A_0$. Let $\tau := \omega_\sigma$, where $cf|R| = \aleph_\sigma$. We assumed that $|R|$ be singular. A well order of A_0 yields, by induction procedure, a τ-sequence $(a_i)_{i<\tau}$ of points of A_0 such the numbers $|(\cdot, a_i)_{A0}|$ are increasing strongly to $|R|$; of course we might assume that for a given strongly increasing τ-sequence of ordinals $\alpha_i \uparrow \tau$ one has

7.3.6. $|(\cdot, a)_{A0}| = \aleph_{\alpha_i+1}$ $(i < \tau)$. In particular, let

7.3.7. $a_{ij} (i < \omega_{\alpha_i+1})$ $(i < \tau)$ be a normal well-order of the set $(7:3:6)_1$.
This being, let us now define a sequence $\sigma_i\,(i<\tau)$ like this: $a_0 = b_0$. Let $0 < i < \omega_\sigma$ and assume that an antichain $B_i := \{a_k)_{k<i}$ be defined; then there exists an index j_0 such that no a_{ij} for $j > j_0$ belongs to $(\cdot, a_k]_S$. In fact, the union of these sets is less than the regular number $|(\cdot, a_j]_S|$. We set $b_j = a_{ij0}$, j_0 being the first ordinal such that no a_{ij} with $j > j_0$ belongs to some $(\cdot, a_k)$ satisfying $k < i$. This being possible for every $i < \tau$, we get the set $B := \{b_i\}_{j<\tau}$. The set B is an antichain of cardinality $\aleph_\sigma$. Now, $b_i < a_i$ for every $i < \tau$ and, in virtue of (7:3:6), $|S_0[b_i, a_i]| = \aleph_{\alpha_i+1}$ $(i<\tau)$. The sets $S_0[b_i, a_i) := B_i$ $(i < \tau)$ are pairwise disjoint; therefore the union Z of these sets has $|R|$ points; but Z is degenerated, because $R_0 Z = B$, $\bigcup_x Z[x, \cdot) = Z$ $(x \in R_0 Z)$ and B is antichain. In other words, R is equinumerous to its own D-subset.

7:3:8. Second case: $|S_0| < |R|$, thus $|S_1| = |R|$, where $S_1 = R \setminus S_0$ and for every point $x \in S_1$ we have $R[x]_{S_1} = |S_1| = |R|$. If some $x \in R$ satisfies $|R(\cdot, x]| = |R|$, the question is settled: R is equinumerous with the chain $R(\cdot, \cdot x]$. Therefore, let us assume that every set $R(\cdot, x)$ has less than $|R|$ members, and consequently:

7:3:9. $S_1[x, \cdot)_R| = |S_1| = |R|$ for every $x \in S_1$. Now, let T be any subtree of S_1 cofinal to S_1. By supposition $|T| < |S_1|$. Let $\aleph_\sigma := cf|T|$; then $\aleph_\sigma$ is regular.

If T contains an antichain A of cardinality $\aleph_\sigma$, then for a strictly increasing ω_τ-sequence k_i of cardinals $\to |R|$ we determine by induction argument a τ-sequence of points $a_i \in A$ and a τ-sequence L_i $(i<\tau)$ of chains L_i such that $a_i < L_i$, $|L_i| \geqslant k_i$; then the union of these sets L_i is a requested D-set of cardinality R.

7:3:10. If every antichain of T is $<\aleph_\sigma$, let us consider the case that T contains a chain $L := (l_i)_{i<\tau}$ of cardinality $|\tau|$; then there is some $i<\tau$ such that $[a_i, \cdot)_{S_1}$ is a chain (in opposite case, $\omega_\sigma := \tau$ being regular let b_0 be any point such that $a_0 < b_0$ and $b_0 \| a_{c_0}$ for some $c_0 < \tau$; for every $o<j<\tau$, we could consider a point b_j such that $a_j < b_j$, $j \geq \sup_{i<j} c_i$ and that $b_j \| c_j$ for some a_j. Then the points $(b_j)_{\xi<\tau}$ should be an antichain of cardinality $\aleph_\sigma$). Consequently, a terminal portion of L should be without ramification in S_1 and therefore for every of its points a we have a chain $[a, \cdot)_{S_1}$ of cardinality $|R|$.

7:3:11. Third case. Every chain and every antichain in T is $<\aleph_\sigma$; the cardinal $|T|=|\tau|$ being regular, the reduction principle says that T contains a chain or an antichain of the cardinality $|\tau|$-contradiction.

The theorem (7:3) is completely proved.

8. Chain-tree alternative for ramifications.

8:1. Theorem. *If a ramified set $(R, \leq)$ of a cardinality $\geq \aleph_0$ contains no tree of cardinality $|R|$, then for every cardinal $k<|R|$ the set $(R \leq)$ contains a chain of cardinality $>k$, and in particular one $>cf|R|$, if $|R|$ is singular; in particular if $|R|$ is infinite and regular, then $(R, \leq)$ is equinumerous to a subtree or to a subchain.*

Proof. The statement is obvious if $|R|$ is regular because if T is a subtree of R cofinal to R we have.

8:2. $R = \cup_t R(\cdot, t]$ $(t \in T)$; now, if by assumption, $|R| > |T|$, then the relation

8:3. $t \in T \Rightarrow |R(\cdot, t]| < |R|$ would be false, because the set R of cardinality $>|T|$ is not a sum of less than $\aleph_\sigma$ of sets $<\aleph_\sigma$ each.

8:4. Therefore, let us consider the case that $|R|$ is singular. Let $k<|R|$. We claim that some $t \in T$ satisfies $|(\cdot, t]_{(R, \leq)}| > k$ and consequently the statement (8:1) holds, because the set $(\cdot, t]_{(R, \leq)}$ is a chain. Now, if every $t \in T$ satisfies $|(\cdot, t]_{(R, \leq)}| \leq k$, then the relation (8:2) would yield $|R| \leq |T| \cdot k < |R| \cdot |R| = |R|$ — contradiction.

9. Some propositions on pseudo-trees.

9.0. In [1935] we have published 12 equivalent tree propositions P_1, $P_2, \ldots, P_{12}$ labelled as postulates; the proof of postulatedness of $P_1, \ldots$ was performed much later; the organic tie between trees and pseudotrees shows that similar propositions and connexions hold true for pseudo-trees as well. Later, we added several new propositions (for the relevant bibliography see. Kurepa Đ. [1977]). We are going to associate to some tree propositions P_k a corresponding proposition P_k^r concerning *ramified sets* or *pseudo-trees* $(R, \leq)$. In this way we have, e. g, following propositions.

9:1. Proposition P_1^r. *Every infinite pseudotree* $(R, \leq)$ *contains a degenerated subset* R_d *of cardinality* $bR := \sup|D|$, $(D \subset R$, *D is degenerated, i. e. for every* $a \in R$ *the cone* $[a]_{(R, \leq)} := \{x \therefore x \in R \wedge x \leq a \vee x \geq a\}$ *is a chain in* $(R, \leq))$.

9:2. Proposition. P_2^r. *Every infinite pseudotree is equinumerous to an own D-subset.*

9:3. Proposition P_3^r. *For every transfinite ramified set system* $(R, \supset)$ *there exists a subsystem of cardinality* $|R|$ *of non-radial elementary segments of* $(R, \supset)$. *For definitions see* 10:2:1.

9:4. Proposition P_4^r. *For every infinite ramified system* $(R, \supset)$ *of sets, the system* $(R^d, \supset)$ *has the same cardinal number as some disjoint subsystem of* R^d *(fundamental proposition on non-overlapping systems of sets).*

9:5. Proposition P_6^r. *If* $(R \leq)$ *is any ramified set that is cofinal to a distinguished ramified sequence, then every natural total ordering* $0(R, \prec)$ *is normal, i. e. there exists a disjoint family of cardinality* $s(R, \prec)$ *of intervals of* $(R, \prec)$.

9:6. Proposition P_7^r. *Let* $(R, \leq)$ *be any ramified set that is cofinal to a distinguished ramified sequence; then every total natural order extension* $(R, \prec)$ *of* $(R, \leq)$ *has a same separability degree.*

9:7. Proposition P_8^r. *Every ramified set* $(R, \leq)$ *cofinal to a distinguished ramified sequence is equinumerous to a subchain or to a subantichain.*

9:8. Proposition P_{16}^r. *(Main pseudotree alternative): Every infinite pseudotree of regular cardinality is equinumerous to an own subchain or to an own subantichain.*

9:9. Proposition P_{17}^r. **(General pseudotree alternative):** *Every infinite pseudotree R contains a chain of cardinality* $|R|$ *or an antichain of a cardinality* $cf|R|$.

9:10. Proposition P_{17}^{r-} *(dual of* P_{17}^r*). Every infinite pseudotree R contains an antichain of cardinality* $|R|$ *or a chain of cardinality* $cf|R|$.

9:11. **Main theorem.**

The propositions

$P_1, P_1^r, P_2, P_2^r, P_3, P_3^r, P_4, P_4^r, P_5, P_6, P_6^r, P_7, P_7^r, P_8, P_8^r, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, P_{16}^r, P_{17}, P_{17}^r, P_{17}^-, P_{17}^{r-}$ *are pairwise equivalent.*

The theorem is a consequence of the theorem 7:10 in Kurepa Đ. [1977] and of the theorems 7:3, 8:1 of the present paper.

10. Some definitions and some theorems.

10:1. Let $(E, \leq)$ be any ordered set. We define *elementary segments* of $(E, \leq)$ as following subsets:

10:1:1. Singletons $\{a\}$ provided $a \in E$ and $(\cdot, a]_{(E, \leq)} = \{a\}$ or $[a, \cdot)_{(E, \leq)} = \{a\}$;

10:1:2. Segments $[a<b]_{(E, \leq)} := \{x \therefore x \in E \wedge a \leq x \leq b\}$.

10:2. **Number b'**. Let $b'(E, \leq) := \sup_X |X|$, X being a system of non-radial elementary segments of $(E, \leq)$.

10:2:1. A system $S \subset P(E)$ is *non-radial* provided for every $(X, Y) \in S^2$ one has either $X \| Y$ (in the sense that $x \in X \wedge y \in Y \Rightarrow x \| y$ or $X \leq Y$ or $Y \leq X$. In particular, for every ramified set $(R, \leq)$ we have the number $b'(R, \leq)$.

If $(\mathcal{R}, \supset)$ is a non overlapping set system, then the number $b'\mathcal{R}$ has a more advantageous definition.

10:3. **Operator $F \to F^d$**. For any system F of sets let F^d be the system of all members of F and of the sets of the form $X \setminus Y$ where $X, Y \in F$. Then we have the following.

10:4. **Theorem.** *For any infinite family F of non overlapping sets one has $b'F = c(F^d)$, where $c(F^d)$ denotes the supremum of the cardinalities of disjoint families of sets belonging to F^d* (cf. Đ. Kurepa [1935] p. 110).

The proof of 10:4. rests on the following

10:5. Lemma. *If (A, B, C, D) is any oriented quadruple of non empty non overlapping sets, then*

(1) $A \setminus B = C \setminus D := X \neq \varnothing$ implies

(2) $A = C \wedge B = D$.

Proof of the lemma 10:5. At first $A \supset \neq B$, $C \supset \neq D$. Further, $A \supset X \wedge \wedge C \supset X \wedge X \neq \varnothing$ imply $A \supset C \vee C \supset A$. We claim that $A = C$. Assume

(3) $A \supset \neq C$.

Then one should have

(4) $A = B \cup X = B \cup (C \setminus D)$

$C = D \cup X = D \cup (A \setminus B)$.

By assumption $A \supset C$ we have

(5) $B \cup (C \setminus D) \supset D \cup (A \setminus B)$ and consequently

$B \supset D$

Now, $C \supset D$; this relation jointly with (5) implies

(6) $B \cap C \supset D$.

1. By assumption $D \neq \varnothing$. Then the sets B, C should be non empty and comparable: One has $B \supset C \vee B \subset C$.

1 : 1. Let $B \supset C$; thus $A \backslash B \subset A \backslash C$, therefore

$$(A \backslash B) \cap X \subset (A \backslash C) \cap X \text{ i. e. (because } C \supset X)$$

$$X \subset \varnothing \text{ — absurdity.}$$

1 : 2. Let $B \subset C$. Then in virtue of (6) one should have $B \supset D$ and consequently $A \supset \neq C$ (assumption) $\supset B \supset D$. Therefore, no point $a \in A \backslash C$. although (7) $a \in A \backslash B (= X)$ should satisfy $a \in C \backslash D$, in contradiction with the assumption (3) and the relation (7). Analogously, one proves that the relation $C \supset \neq A$ is not possible (it is sufficient to apply the result of the case 3) for the sequence (C, B, A, D). The example of the quadruple $(\{1, 2, 3\}, \{1, 2\}, \{3\}, \varnothing)$ shows that $\{1, 2, 3\} \backslash \{1, 2\} = \{3\} \backslash \varnothing$, without that conditions (2) be satisfied; consequently, the lemma 10: 5 need not hold for the case that some term in (A, B, C, D) be the empty set.

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Ovaj rad je finansirala Zajednica za nauku SR Srbije preko Matematičkog instituta Beograd.

Address:
Kurepa, Đuro
Zagrebačka 7
Beograd, Yugoslavia