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PREVENTIVE REPLACEMENT POLICIES UP TO THE FIRST FAILURE

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Introduction

Consider some equipment whose interfailure time is a random variable. We want to make this time as long as possible. One way to do that is "preventive replacements". There is more than one model describing them, as it can be found in [1]. [2]. Almost every paper on this topic considers a different model of replacements. This paper also introduces a new model. In order to prevent failure of equipment we replace still working overaged equipment or stop the whole process. If working equipment failes we also stop the process. We want to determine the plan of replacements and stopping polices as a function of given parameters in order to realize maximal profit (minimal loss).

This paper consists of three parts. The first part describes the problem. In the second part we derive basic functional equations and prove the theorem of existence of optimal polices. In the third part some numerical problems of determinating optimal plan are discussed.

1. Let $X_1, X_2, \ldots$ be a sequence of nonnegative independent identically distributed random variables with distribution function $F(x)=P(X_i<x)$, $F(0)=0$. Let $t, T_1, T_2, \ldots, T_k$, $1 \leqslant k \leqslant \infty$ be a sequence of positive real numbers such that $S_k=\sum_{i}^{k} T_j \leqslant t$. Let a, b, c be constants and $a>0$, $b>0$, $c \geqslant 0$. Define the random variable in this way

$$(1) \qquad Y=(aX_1-c)\ I\{X_1<T_1\}+\sum_{i=1}^{k-1}[a(S_i+X_{i+1})-ib-c]\times$$

$$\times I\{X_1 \geqslant T_1, \ldots, X_i \geqslant T_i, X_{i+1}<T_{i+1}\}+[aS_k-(k-1)b]\ I\{X_1 \geqslant T_1, \ldots, X_k \geqslant T_k\},$$

$$Y=0, \quad T_1=0 \text{ or } t=0.$$

We interpret introduced objects in the following way. $X_1, X_2, \ldots$ are interfailure times of the first, second,... equipment respectively. At the moment o we put on the first equipment. At the moment S_{i+1} we put out of the operation i-th equipment and put into operation $i+1$-th, if i-th equipment did not fail before the moment S_{i+1}. Replacements are made up to the first failure. or stopped at the moment S_k. At the moment t we stop anyway. Constants a, b, c are profit per unit of interfailure time, cost of a planned replacement and loss due to failure before the moment of planned replacement respectively. Variable Y is, then the total operation profit due to the work of the whole system. $T_1=0$ means that we do not let the system work.

According (1) we have

$$k(t, T_1, \ldots, T_k) = EY = \int_0^{T_1} (ax-c)\, dF(x) + \tag{2}$$

$$+\sum_{i=1}^{k-1} \int_0^{T_{i+1}} [a(S_i+x) - ib - c]\, dF(x) \prod_{j=1}^{i} (1-F(T_j)) +$$

$$+[a S_k - (k-1)\, b] \prod_{j=1}^{k} (1-F(T_j)),$$

$$k(t, T) = k_0(T) = a \int_0^T [1-F(x)]\, dx - cF(T), \quad 0 \leqslant T \leqslant t, \quad \text{for } k=1. \tag{3}$$

2. Call the sequence $T_1, T_2, \ldots$ the plan and denote it by (T). Let, particularly, $(T)_j = (T_1, T_2, \ldots, T_j)$. Let

$$k(t) = \sup_{(T)} k(t, (T)), \quad k_i(t) = \sup_{(T)_j\, 1\leqslant j\leqslant i} k(t, (T)_j), \quad i=1, 2, \ldots. \tag{4}$$

We want to find the plan $(T)^*$ for which we obtain supremum in (4). In general (2) is not convenient though we may get certain results (in case of differentiability).

Let the plan (T) suggest at least one replacement i.e. $(T) = (T_1, T_2, \ldots)$ and let $(T') = (T_2, \ldots)$. Then

$$\begin{aligned} k(t, (T)) &= k(t, (T);\ X_1 < T_1) + k(t, (T);\ X_1 \geqslant T_1) = \\ &= E(aX_1 - c;\ X_1 < T_1) + [a T_1 - b + k(t-T_1, (T'))]\ (1-F(T_1)) = \\ &= k_0(T_1) + [k(t-T_1, (T')) - b]\ (1-F(T_1)) = \\ &= \bar{k}(T_1) + k(t-T_1, (T'))\, B(T_1), \end{aligned} \tag{5}$$

where

$$\bar{k}(T_1) = k_0(T_1) - b(1-F(T_1)), \quad B(T_1) = 1-F(T_1). \tag{6}$$

For $i \geqslant 1$ it follows that

$$k_1(t) = \sup_{0\leqslant T_1\leqslant t} k(t, T_1) = \sup_{0\leqslant T_1\leqslant t} k_0(T_1), \tag{7}$$

$$k_{i+1}(t) = \max\{k_1(t), \sup_{(T)_j,\, 2\leqslant j\leqslant i+1} k(t, (T)_j)\} =$$

$$= \max\{k_1(t), \sup_{0<T_1<t} [\bar{k}(T_1) + \sup_{(T')_j,\, 1\leqslant j\leqslant i} k(t-T_1, (T')_j)\, B(T_1)]\},$$

or

$$k_{i+1}(t) = \max\{k_1(t), \sup_{0<T<t} [\bar{k}(T) + k_i(t-T)\, B(T)]\}. \tag{8}$$

In the same manner

(9) $$k(t)=\max\{k_1(t), \sup_{0<T<t}[\bar{k}(T)+k(t-T)\ B(T)]\}.$$

By definition in (4)

(10) $$k_1(t)\leqslant k_i(t)\leqslant k_{i+1}(t)\leqslant k(t)\leqslant at, \qquad i=1, 2, \ldots.$$

But, we have also

Lemma 1. $k(t)=\lim_{i\to\infty} k_i(t)$

Proof: Let (T) be an infinite plan, and let $t'=\lim S_i\leqslant t$. Let Z be non failure operation time up to the first failure. Then for $i\geqslant 1$ and $(T^i)=(T_{i+1}, \ldots)$ we have

$$k(t, (T))=k(t, (T);\ Z<S_i)+k(t, (T);\ Z\geqslant S_i)=$$
$$=k(t, (T));\ Z<S_i)+[a S_i-ib+k(t-S_i, (T^i))]\ P(Z\geqslant S_i)=$$
$$=k(t, T_1, \ldots, T_i,\ t-S_i)+[k(t-S_i, (T^i))-k_0(t-S_i)]\ P(Z\geqslant S_i)\leqslant$$
$$\leqslant k(t, T_1, \ldots, T_i,\ t-S_i)+[a(t-S_i)+c]\ P(Z\geqslant S_i).$$

Let $P(Z\geqslant t')=0$, otherwise it would be $k(t, (T))=-\infty$. Then $P(Z\geqslant S_i)\to 0$, $i\to\infty$ and $[a(t-S_i)+c]\ P(Z\geqslant S_i)\leqslant\varepsilon$, for i sufficiently large i. e. the finite plan always exists which is good enough.

Lemma 2. $0\leqslant k_i(t+\varepsilon)-k_i(t)\leqslant a\varepsilon$, $i\geqslant 0$, $t\geqslant 0$. $0\leqslant k(t+\varepsilon)-k(t)\leqslant a\varepsilon$, $t\geqslant 0$.

Proof: Plan (T) on $[0, t]$ is also a plan on $[0, t+\varepsilon]$ because $\sum_i^k T_i\leqslant t<t+\varepsilon$. Then $k(t, (T))=k(t+\varepsilon, (T))$. Hence, the left inequality is proved. If (T) is a plan on $[0, t+\varepsilon]$ it can be reduced to a plan $(T)^t$ on $[0, t]$ stopping at t. By (10), $k(\varepsilon)\leqslant a\varepsilon$, so obviously $k(t+\varepsilon, (T))\leqslant k(t, (T)^t)+a\varepsilon$. From this we have proved the right inequality.

Consequence: Func ions $k_i(t)$, $i=1, 2, \ldots, k(t)$ are continuous and $k_i(t)\to k(t)$, $i\to\infty$, uniformly with respect to t on finite intervals.

Lemma 3. $k(t)=k_1(t)$, $0\leqslant t\leqslant t_0$,

$$t_0=\sup\{t: k(u)=k_1(u), \text{ for all } u,\ 0<u\leqslant t\}=$$
$$=\sup\{t: \bar{k}(T)+k_1(u-T)\ B(T)\leqslant k_1(u),\ 0<T<u\leqslant t\}\geqslant\frac{b}{a}.$$

Proof: $k(t)=k_1(t)$, $t\leqslant t_0$ means that for T, $0<T<t$, follows $k_1(t)\geqslant\bar{k}(T)+k(t-T)\ B(T)=\bar{k}(T)+k_1(t-T)\ B(T)$. Suppose the opposite. Then considering (8) $k_2(u)=k_1(u)$ i.e. $\bar{k}(T)+k_2(u-T)\ B(T)\leqslant k_1(u)$, $0<T<u$ i.e.

$k_3(u)=k_1(u)$ etc. $k_i(u)=k(u)=k_1(u)$, $i=2, 3, \ldots$. If $k_1(\delta-T)-b\leqslant 0$ or $k_1(\delta)\leqslant b$ then from $\bar{k}(T)+k_1(\delta-T)\;\;B(T)=k_0(T)+[k_1(\delta-T)-b]\;\;B(T)\leqslant\leqslant k_0(T)\leqslant k_1(\delta)$, $0<T<\delta$

$$(11)\qquad t_0\geqslant\delta_0=\sup\{\delta: k_1(\delta)\leqslant b\},$$

follows because k_1 is an increasing function in δ.

By definition in (7), if $k_1(\delta)\leqslant b\;\; a\int_0^T[1-F(x)]\,dx-CF(T)\leqslant b,\;\; T\leqslant\delta$, follows. The last relation always holds if $a\int_0^T[1-F(x)]\,dx\leqslant b$, i.e. $t_0\geqslant\delta_0\geqslant T_0=$

$$=\sup\left\{T: \int_0^T[1-F(x)]\,dx\leqslant\frac{b}{a}\right\}\geqslant\frac{b}{a}.$$

If $\mu=\int_0^\infty[1-F(x)]\,dx\leqslant\frac{b}{a}$, i.e. $T_0=\infty$ we, of course, do not need any preventive replacement.

Let $F(x)$ be a continuous function. Then $k_0(T)$ and $k(T,t)=\bar{k}(T)+ +k(t-T)\;B(T)$, $0<T<t$, are also continuous and there exists T', $0\leqslant T'\leqslant t$, for which $k_1(t)=k_0(T')$. If $k(t)>k_1(t)$ there exists T'' for which $k(T'',t)= =\sup\limits_{0<T<t}k(T,t)$. In the first case optimal plan is $(T)=(T')$. In the second case $T''=T_1$ is the moment of the first replacement. Further, we examine $k(t-T'')$ in the similar manner and obtain optimal finite plan, as the theorem states.

Theorem: *Let $F(x)$ be a continuous function. Then there exists a finite plan for which $k(t)=k(t,(T))$.*

Proof: Suppose that following the previous costruction we obtain infinite plan. Then for every $i\geqslant 1$

$$k(t)=\sum_{j=0}^{i-1}E(aZ-ib-c;\; S_j\leqslant Z<S_{j+1})+[a\,S_i-ib+k(t-S_i)]\;\;P(Z\geqslant S_i),$$

(proved by induction). Let, as in *Lemma 1.* $t'=\lim S_i\leqslant t$ and $P(Z\geqslant t')=0$, hence $k(t-S_i)=k(t'-S_i)$. For i s.l. $t'-S_i\leqslant\frac{b}{a}$ and by *Lemma 3.* $k(t'-S_i)= =k_1(t'-S_i)$ which means that in the interval $[t'-S_i, t']$ we do not make any replacement but we stop the process, which is in contradiction with the assumption of infinite plan.

3. In order to obtain optimal plan we have to find the function $k(t)$ using equation in (9) or reccurence formula in (8). Because $k(t)$ is calculated using $k(t-T)$, $0<T<t$, it is necessary to have initial values for small t, which we have obtained from Lemma 3.

Particularly, for decreasing failure rate distribution we do not need to plan replacements i.e. we stop at the moment T' *for which* $k(t)=k_1(t)=\sup_{0\leqslant T\leqslant t} k_0(t)=$ $=k_0(T')$.

Equation in (9) can be written in the form

$$k(t)=\max\{k_1(t), \sup_{0<T<t}[\bar{k}(t-T)+k(T)\ B(t-T)]\}. \tag{12}$$

The first, we need to find t_0 if it is possible. In that case in (9) we do not consider $k_1(t)$ for $t>t_0$. Otherwise, we can choose come other t_1, $\frac{b}{a}\leqslant t_1\leqslant t_0$, and subdivide the interval $[t_1, t]$ with points $t_1<t_2<\cdots<t_n=t$, depending on given tolerance. Then for $j=1, 2, \ldots, n$ we calculate values

$$\tilde{k}(t_j)=\max\{k_1(t_j), \max_{1\leqslant l\leqslant j-1}[\bar{\kappa}(t_j-t_l)+\tilde{\kappa}(t_l)\ B(t_j-t_l)]\}, \tag{12'}$$

and obtain the sequence $\tilde{\kappa}(t_1), \tilde{k}(t_2), \ldots, \tilde{k}(t_j)$. Then we take $k(t)=\tilde{k}(t_n)$.

When we find $k(t)$, we have to calculate corresponding sequence which gives us the optimal plan. In order to do that we start from the end i.e. we look for the value $t_n^*=t_l$, which gives us maximum in (12′) for $j=n$. Further, we look for value t_{n-1}^* which gives maximum for $\tilde{\kappa}(t_n^*)$ and so on, up to do certain i for which we have $\tilde{k}(t_i^*)=k_1(t_i^*)$. We also have to calculate the value t_0^*, for which $k_1(t_i^*)=k_0(t_0^*)$. Then, optimal plan is $(t_n-t_n^*, t_n^*-t_{n-1}^*, \ldots, t_{i+1}^*-t_i^*, t_0^*)$.

Example: Let $F(x)=x$, $0\leqslant x\leqslant 1$. Let $a=1$, $b=0{,}12$, $c=0{,}5$. Let $t=1$. We have

$$\kappa_0(t)=(1-t)\ t\cdot 0{,}5, \quad t\leqslant 1; \quad k_1(t)=\begin{cases}0{,}125 & 0{,}5<t\leqslant 1\\ (1-t)\ t\cdot 0{,}5, & t\leqslant 0{,}5\end{cases}$$

$$\bar{k}(t)=(-0{,}5\ t+0{,}62)\ t-0{,}12, \quad 0\leqslant t\leqslant 1,$$

$$\tilde{k}(t_j)=\max\{k_1(t_j), \max_{1\leqslant l\leqslant j-1}\{[(-0{,}5)\ (t_j-t_l)+0{,}62-\kappa(t_l)]\ (t_j-t_l)-0{,}12+\tilde{k}(t_l)\}\}.$$

Using (11) we obtain $\delta_0=0{,}4$. We subdivide the interval [0,4; 1] into equal parts of lenght 0,05 and step by step we obtain

$\tilde{k}(0{,}4)=k_1(0{,}4)=0{,}12$

$\tilde{k}(0{,}45)=k_1(0{,}45)=0{,}12375$

$\tilde{k}(0{,}5)=\tilde{k}(0{,}55)=\cdots=\tilde{k}(0{,}85)=k_1(0{,}5)=0{,}125$

$\tilde{k}(0{,}9)=0{,}1258125$ obtained for $t_l=0{,}45$

$\tilde{k}(0{,}95)=0{,}126875$ obtained for $t_l=0{,}45$

$\tilde{k}(1{,}00)=0{,}1275$ obtained for $t_l=0{,}5$.

Hence, the optimal plan is (0,5; 0,5). Obviously a better approximation of the starting value would shorten the calculation.

In certain circumstances we may obtain the result using differential calculus. Let the function F be differentiable on $[0, t]$. Function $k(t, T_1, \ldots, T_k)$ in (1) is constant for $t > T_k$. Therefore, put $k(T_1, \ldots, T_k) = k(t, T_1, \ldots, T_k)$. Then

$$k = k(T_1, \ldots, T_k) = \bar{k}(T_1) + k(T_2, \ldots, T_k)\, B(T_1), \tag{13}$$

and in general

$$k(T_i, \ldots, T_k) = \bar{k}(T_i) + k(T_{i+1}, \ldots, T_k)\, B(T_i), \tag{14}$$
$$1 \leqslant i \leqslant k-1, \quad k(T_k) = k_0(T_k).$$

Let $c > 0$. We can see easily that

$$\frac{\partial k}{\partial T_i} = B(T_0)\, B(T_1) \cdots B(T_{i-1}) \left[\frac{\partial \bar{k}(T_i)}{\partial T_i} + k(T_{i+1}, \ldots, T_k)\frac{\partial B(T_i)}{\partial T_i}\right], \tag{15}$$
$$1 \leqslant i \leqslant k-1$$

$$B(T_0) = 1, \quad \frac{\partial k}{\partial T_k} = \frac{\partial k_0(T_k)}{\partial T_k} B(T_0) \cdots B(T_{k-1}),$$

and if $B(T) = 1 - F(T) > 0$, $T < t$, then the system of equations $\frac{\partial k}{\partial T_i} = 0$, $i = \overline{1, k}$ is equivalent to

$$\frac{\partial \bar{k}(T_i)}{\partial T_i} + k(T_{i+1}, \ldots, T_k)\frac{\partial B(T_i)}{\partial T_i} = 0, \; 1 \leqslant i \leqslant k-1, \; \frac{\partial \bar{k}(T_k)}{\partial T_k} = 0. \tag{16}$$

Using (6), (3), (16) we obtain

$$-\frac{B(T_{i-1})}{B'(T_{i-1})} = \frac{c-b}{a} + \frac{1}{a} k(T_i, \ldots, T_k), \; 2 \leqslant i \leqslant k, \; -\frac{B(T_k)}{B'(T_k)} = \frac{c}{a}, \tag{17}$$

and using (14)

$$-\frac{B(T_{i-1})}{B'(T_{i-1})} = -\frac{b}{a} + \int_0^{T_i} B(x)\, dx - B(T_i)\frac{B(T_i)}{B'(T_i)}, \quad \text{i.e.}$$

$$\frac{1 - F(T_{i-1})}{F'(T_{i-1})} = -\frac{b}{a} + \int_0^{T_i} (1 - F(x))\, dx + (1 - F(T_i))\frac{1 - F(T_i)}{F'(T_i)}, \tag{18}$$
$$2 \leqslant i \leqslant k,$$

$$\frac{1 - F(T_k)}{F'(T_k)} = \frac{c}{a}, \quad T_1 + T_2 + \cdots + T_k \leqslant t.$$

Hence, if optimal solution can be obtained by differentiation, it has to satisfy the system in (18).

As we can see in (18) neither T_k nor $T_{k-1}, T_{k-2}, \ldots$ depend on k. Hence, if the solution of the system (18) we denote considering the order of obtaining by $T_1^*, T_2^*, \ldots (T_1^* = T_k)$, then the optimal plan for a given k is $T_k^*, T_{k-1}^*, \ldots, T_1^*$.

So, using (17) and (14), we have

$$k(T_1, \ldots, T_k) = k(T_k^*, \ldots, T_1^*) = k^*(T_k^*) = \tag{19}$$

$$= a\left[\int_0^{T_k^*} (1-F(x))\, dx + \frac{(1-F(T_k^*))^2}{F'(T_k^*)}\right] - c.$$

Still, it is in question which k is optimal. Let $S_i^* = T_1 + \cdots + T_i = T_i^* + \cdots + T_1^*$. Using *Theorem* and *Lemma 3.* we see that it is necessary to plan replacements as long as $t - S_i^* > t_0$. Let $0 \leqslant t - S_j^* \leqslant t_0 < t - S_{j-1}^*$. It means that $k(t-S_j^*) = k_1(t-S_j^*)$. If $k_1(t-S_j^*) = 0$ it means that the process stops at S_j^*, and we have that optimal plan with $j-1$ replacements is $T_j^*, T_{j-1}^*, \ldots, T_1^*$. If $k_1(t-S_j^*) > 0$, it means that j replacements is necessary. Optimal plan using j replacements is $T_{j+1}^*, T_j^*, \ldots, T_1^*$, On the other hand, considering that optimal plan is finite, and comparing related $k^*(T_1^*)$, $k^*(T_2^*), \ldots,$ we can also find the optimal lenght of the plan.

Example: Let $F(x) = x$, $0 < x \leqslant 1$, $t \leqslant 1$. System (18) becomes

$$\theta_{i+1}^* = \frac{1}{2}\theta_i^* + \frac{1}{2} - \frac{b}{a}, \quad \theta_i^* = 1 - T_i^*, \quad i = 1, 2, \ldots, \quad \theta_1^* = \frac{c}{a}.$$

In order to find the solution it has to be:

If $\theta_2^* - \theta_1^* = \frac{1}{2}\left(\frac{c}{a}\right)^2 + \frac{1}{2} - \frac{b}{a} - \frac{c}{a} > 0$, i.e. $1 - t < \frac{c}{a} < 1 - \sqrt{2\frac{b}{a}}$, it will be $\theta_{i+1}^* - \theta_i^* = \frac{1}{2}(\theta_i^{*2} - \theta_{i-1}^{*2}) > 0$, $i = 2, 3, \ldots$.

Then, using (19), we have

$$k^*(T_{i+1}^*) - k^*(T_i^*) = \frac{a}{2}[(1-T_{i+1}^*)^2 - (1-T_i^*)^2] - \frac{a}{2}[\theta_{i+1}^{*2} - \theta_i^{*2}] > 0,$$

and the optimal lenght of the plan is k for which $T_1^* + \cdots + T_k^* \leqslant t < T_1^* + \cdots + T_{k+1}^*$.

If $\theta_2^* - \theta_1^* < 0$ i.e. $\max\left\{1-t,\ 1-\sqrt{2\frac{b}{a}}\right\} < \frac{c}{a} < 1$, then $\theta_{i+1}^* - \theta_i^* < 0$ i.e. $k^*(T_{i+1}^*) - k^*(T_i^*) < 0$ and then $k = 1$ is optimal, and $T^* = 1 - \frac{c}{a}$.

In the previous numerical example we had $t = 1$, $\frac{c}{a} = 0{,}5$ $\frac{b}{a} = 0{,}12$, which satisfy the condition $1 - t < \frac{c}{a} < 1 - \sqrt{2\frac{b}{a}}$. Then $T_1^* = 1 - \frac{c}{a} = 0{,}5$, $T_2^* = 1 - \left(\frac{1}{2}0{,}5^2 + \frac{1}{2} - 0{,}12\right) = 0{,}495$, $T_3^* = 0{,}4924875$, and then we have $T_1^* + T_2^* < 1 < T_1^* + T_2^* + T_3^*$. It means $(0{,}495;\ 0{,}5)$ is the exact optimal plan, and it does not differ too much from the plan obtained by approximative calculation. Using (19) we have $k^{*}(T_2^{*})=k(T_2^{*}, T_1^{*})=0{,}1275125=k(1)$, but using approximative calculation we have $\tilde{k}(1)=0{,}1275$.

If $c=0$, it is obvious that we should not stop the process before the moment t i.e. $T_1+T_2+\cdots+T_k=t$. But in this case we cannot obtain T_k by derivation as we have done in (18). Instead of that we have to solve the system of equations with respect to $T_1, T_2, \ldots, T_{k-1}$ with condition $T_k=t-T_1-\cdots-T_{k-1}>0$. We can do that by giving the value of T_k and testing whether that value is optimal, using (19). This procedure does not seem to be convenient in general.

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